Extreme Value Theory to Estimating Value at Risk

Turan (2003) An Extreme Value Approach to Estimating Volatility and Value at Risk, *J of Business*
Gomes and Pestina (2007) A Sturdy Reduced-Bias Extreme Quantile Estimator, *JASA*
Davison and Smith (1990) Models for exceedance over high thresholds, *JRSSB*

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1. Background

In many areas of application, such as statistical quality control, insurance, and finance, a typical requirement is to estimate a high quantile, that is, the ***Value at Risk*** at a level p(***VaRp***), high enough so that the chance of an exceedance of that value is equal to p, small. A VaR model measures market risk by determining how much the value of the portfolio would fall given the probability over the time span. The most commonly used VaR models assume the probability distribution of the daily changes in market variable is normal, an assumption that is far from perfect. The changes in many variables exhibit significant amount of skewness and kurtosis. Turan (2003) studies the level of short rate changes and its volatility at the extreme tails of the distribution. The tails of the empirical distribution appear to be thicker than the tails of the normal distribution. The paper emphasized that the extreme value theory (EVT) provides a more accurate estimate of the rate of occurrence and volatility of extreme observations, thus VaR calculation are more precise and robust in terms of risk management. The theoretical framework using EVT has been derived using interest rate changes as an example. Compared with environment research using EVT like Davison & Smith (1990), methods like Maximum Likelihood Estimate (GEV and possion-GPD) and Least Square have been used, but diagnostics leaves much room to be improved. Gomes & Pestina (2007) proposed a Bias-Reduction Quantile (VaR) estimator which could reduce the high bias for the low thresholds, especially targeted for modeling and estimation of financial time series.

Let $X_{max,n}$ denotes the maximum daily interest rate changes. $X_{max,n} = \max(X_1, X_2, \ldots, X_n)$, To find the limiting distribution for maxima $H_{max}(x)$, the **GEV**(Generalized Extreme Value) distribution

$$H_{\max,\xi}(x;\mu,\sigma) = \exp\left\{-\left[1+\xi_{\max}\left(\frac{X_{\max}-\mu_{\max}}{\sigma_{\max}}\right)\right]^{-1/\xi_{\max}}\right\}; \quad 1+\xi_{\max}\left(\frac{X_{\max}-\mu_{\max}}{\sigma_{\max}}\right) \geq 0$$

with μ location parameter, σ scale parameter,
ξ >0 Frechet, fat tailed; ξ <0 Weibull; ξ =0 Gumbel

Let y_t denotes an exceedance over threshold u. $$F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)}.$$

The **GPD**(Generalize Pareto Distribution)

$$F_u(y) \approx G(y; \sigma_u, \xi) = 1 - \left(1 + \xi\frac{y}{\sigma}\right)_+^{-1/\xi}.$$

$\xi > 0$ Pareto, long tail; $\xi = 0$ Exponential; $\xi < 0$ Uniform, short tail;

The author uses both GEV and the Poisson-GPD approach. It is time homogeneous case, i.e. the location and scale parameters are time independent. The methods used are

1. Maximum Likelihood
2. Least Square

The MLE methods are to choose the parameters (μ, ψ, ξ) or (λ, σ, ξ), similar to Davison & Smith's approach. The diagnostics uses Likelihood ratio test to compare among different distributions.

2. Data and Model Assessment

The data sets used in this study consist of daily observations for the annualized yields on 3-month, 6-month, 1-year, and 10-year U.S. Treasury securities from 1954 to 1998. We set the threshold to be 2% of the right and left tails of the distribution. The results from MLE methods of GEV vs. Gumbel are compared, the parameter estimates are given, and the Gumbel distribution is strongly rejected. The MLE methods of generalize Pareto vs. exponential are also conducted, and Pareto distribution is supported due to long tailed.

Overall, the extreme value theorem methods support the *fat tailed* distribution of the extreme of the interest rate changes, and give good estimates of the parameters (μ, ψ, ξ) or (λ, σ, ξ), which could be very helpful in calculating the VaR risk factor. Another important result is that extreme value theory does not assume the underlying distribution to be symmetric, therefore two extremes, i.e. maxima and minima could vary in distribution.

3. Extreme Value VaR Calculation

The critical value ψ that corresponds to various levels of α can be estimated by:

$$\Psi_{GPD} = \mu + \left(\frac{\sigma}{\xi}\right)\left[\left(\frac{\alpha N}{n}\right)^{-\xi} - 1\right].$$

$$\Psi_{GEV} = \mu + \left(\frac{\sigma}{\xi}\right)\left[\left(-\ln\left(1 - \frac{\alpha N}{n}\right)\right)^{-\xi} - 1\right].$$

And the VaR risk factor is thus estimated by: $\mathrm{VaR}(A, \alpha) = A_S \Psi.$

The results show that the ψ_{GPD} and ψ_{GEV} are up to 38% higher than the threshold given by standard normal distribution, which means the thresholds are further into the tail distribution than normal due to its "fat tail". The volatility σ is higher for the occurrence rate of maxima than for minima. To do the validation, the paper uses **MCMC** to simulate the short rate changes, and then compare the extreme value prediction with the simulated rates. EVT provides robust estimates.

4. A Sturdy Reduced-Bias Extreme Estimator

Gomes and Pestina (2007) studies the *high bias* for low thresholds, that is, for large number of top order-statistics used for simulation. The paper explores the bias-reduction techniques for heavy tail distribution through bias-corrected *tail index estimator* (Hill estimator). The theoretical derivation is given and the asymptotic properties are provided. The authors claimed that it provides better extreme value estimator.

5. Conclusion

The main POT methods for environmental research have been utilized, i.e. possion-GPD, maximum likelihood, as well as least square regression. The author did not do multiple thresholds comparison. Therefore the diagnostic part is less persuasive. The Frechet and Pareto heavy tailed Distribution are strongly supported for the rate of occurrence for financial extremes, interest rate changes as an example. And extreme value theory provides accurate and robust measure of VaR calculation.