More on Detection and Attribution

- Levine and Berliner (*J. Clim* **12**, 564–574, 1999)
- Levine, Berliner and Shea (*J. Clim* **13**, 3805–3820, 2000)
- Berliner and Kim (*J. Clim* **21**, 1891–1910, 2008)
- Lopez, Tebaldi, New, Stainforth, Allen and Kettleborough (*J. Clim* **19**, 4785–4796, 2006)

Optimal Signal Detection

This is based on Levine's and Berliner's (*J. Clim* **12**, 564–574, 1999) reinterpretation of a series of papers by Hasselman, e.g. *J. Clim* **6**, 1957–1971 (1993) or *Climate Dynamics* **13**, 601–611 (1997).

Suppose the observed climate signal $\mathbf{\Psi}$ satisfies

$$\mathbf{\Psi} \;=\; \mathbf{\Psi}^S + \tilde{\mathbf{\Psi}}$$

interpreted as "signal+noise". In practice, we usually assume both $\mathbf{\Psi}$ and $\mathbf{\Psi}^S$ are in fact *anomalies* from some reference time period. We also assume

- $\mathbf{\Psi}^S = \sum_{i=1}^{p} a_i \mathbf{g}_i$ where $\mathbf{g}_1, ..., \mathbf{g}_p$ are p known signal patterns and $a_1, ..., a_p$ are unknown weights; also write $\mathbf{\Psi}^S = G\mathbf{a}$.
- $\tilde{\mathbf{\Psi}}$ is a vector of "errors" with mean 0 and covariance matrix C.

Optimal fingerprints (Hasselmann)

- $d_i = \mathbf{f}_i^T \mathbf{\Psi}$ is "detector" (and $\mathbf{f}_i$ is "fingerprint")
- $\mathbf{d}^S = (d_1^S, ..., d_p^S) = (\mathbf{f}_1^T \mathbf{\Psi}, ..., \mathbf{f}_p^T \mathbf{\Psi})$
- Fingerprints $\mathbf{f}_i$ constructed to maximize signal to noise ratio

$$\rho^2(\mathbf{d}^S) \;=\; (\mathbf{d}^S)^T D^{-1} \mathbf{d}^S,$$

  (i, j) entry of D is $\mathrm{Cov}(\tilde{d}_i, \tilde{d}_j) = \mathrm{Cov}(\mathbf{f}_i^T \tilde{\mathbf{\Psi}}, \mathbf{f}_j^T \tilde{\mathbf{\Psi}}) = \mathbf{f}_i^T C \mathbf{f}_j$.

- This optimization problem leads to $\mathbf{f}_i^* = C^{-1} \mathbf{g}_i$, and hence $\mathbf{d}^* = G^T C^{-1} \mathbf{\Psi}$.
- Statistical significance of the signal determined through $\rho^2(\mathbf{d}^*)$.

Alternative formulation (Levine & Berliner)

- Regression equation

$$\mathbf{\Psi} \;=\; G\mathbf{a} + \tilde{\mathbf{\Psi}}$$

- GLS estimates $\hat{\mathbf{a}} = (G^T C^{-1} G)^{-1} G^T C^{-1} \mathbf{\Psi}$ and hence $\hat{\mathbf{\Psi}}^S = G\hat{\mathbf{a}}$.

- Under Gaussian assumptions, $\hat{\mathbf{a}} \sim N[\mathbf{a}, (G^T C^{-1} G)^{-1}]$.

- Test $H_0 : \; \mathbf{a} = 0$ against $H_a : \; \mathbf{a} \neq 0$: UMPI test of level α rejects H_0 if

$$T \;=\; \mathbf{\Psi}^T C^{-1} G (G^T C^{-1} G)^{-1} G^T C^{-1} \mathbf{\Psi} \;>\; \chi^2_p(1-\alpha).$$

- But $T = \mathbf{\Psi}^T \mathbf{f}^* (G^T C^{-1} G)^{-1} (\mathbf{f}^*)^T \mathbf{\Psi} = \rho^2(\mathbf{d}^*)$. Therefore, the two tests are equivalent.

If C is unknown...

Use a sample estimate from historical data, but the χ^2_p distribution may be a poor approximation unless sample size n is very large.

Question: Is there an exact result, analogous to Hotelling's T^2?

Attribution

Suppose we reject the null hypothesis of the previous section, that there is no climate change signal above baseline. We cannot immediately attribute the signal to any of $\mathbf{g}_1, ..., \mathbf{g}_p$. Instead:

- Two estimates — $\hat{\mathbf{a}}_M$ and $\hat{\mathbf{a}}_{\text{obs}}$ from climate model and observations
- Assume distributions $\hat{\mathbf{a}}_M \sim N[\mathbf{a}_M, C_M]$ and $\hat{\mathbf{a}}_{\text{obs}} \sim N[\mathbf{a}, C_{\text{obs}}]$. (N.B. $C_{\text{obs}} = (G^T C^{-1} G)^{-1}$.)
- Test $H_0 : \ \mathbf{a} = \mathbf{a}_M$ against $H_0 : \ \mathbf{a} \neq \mathbf{a}_M$
- Under H_0 we have $\mathbf{a}_{\text{obs}} - \mathbf{a}_M \sim N[0, C_{\text{obs}} + C_M]$.
- Hence reject H_0 if

$$(\mathbf{a}_{\text{obs}} - \mathbf{a}_M)^T (C_{\text{obs}} + C_M)^{-1} (\mathbf{a}_{\text{obs}} - \mathbf{a}_M) \ > \ \chi^2_p(1-\alpha).$$

Issues

- C and C_M unknown — estimate from climate model runs, then construct consistency test using Hotelling's T^2 distribution (but not clear exactly how this works)

- Null and alternative hypotheses the wrong way round??? Usually one tries to control the probability of falsely rejecting the null hypothesis, but that suggests the null and alternative should be interchanged

- Test is usually performed *only* if initial "detection" test rejects $\mathbf{a} = 0$. But that makes it harded to asses true significance level.

Alternatives

1. Analogy with *bioequivalence* — test $H_0 : \mathbf{a} \neq \mathbf{a}_M$ against $H_a : \mathbf{a} = \mathbf{a}_M$. Idea of Brown, Casella and Hwang (*JASA* 1995). Define $\boldsymbol{\theta} = \mathbf{a} - \mathbf{a}_M$, $\hat{\boldsymbol{\theta}} = \hat{\mathbf{a}} - \hat{\mathbf{a}}_M$, $\Sigma = C_{\text{obs}} + C_M$. If $\hat{\boldsymbol{\theta}} \sim N[\boldsymbol{\theta}, \Sigma]$ then H_0 is rejected if and only if the confidence set

$$\left\{ \boldsymbol{\theta} : \ (\boldsymbol{\theta}^T \Sigma^{-1} \boldsymbol{\theta})^{1/2} \leq z_\alpha + (\boldsymbol{\theta}^T \Sigma^{-1} \boldsymbol{\theta})^{-1/2} (\hat{\boldsymbol{\theta}}^T \Sigma^{-1} \boldsymbol{\theta}) \right\}$$

does not contain 0. (Need approximations for Σ unknown.)

2. Simple v. simple test: $H_0 : \mathbf{a} = 0$ versus $H_a : \mathbf{a} = \mathbf{a}_M$. For C known, most powerful α-level test rejects H_0 when

$$T_M \ = \ \frac{\mathbf{a}_M^T C_{\text{obs}}^{-1} \hat{\mathbf{a}}_{\text{obs}}}{(\mathbf{a}_M^T C_{\text{obs}}^{-1} \mathbf{a}_M)^{1/2}} \ > \ z_\alpha .$$

Other general points about these procedures

- **Covariance Estimation.** Many difficulties in practice — often use long-run control model runs to estimate C but even this may not be adequate. (*Note* — subsequent to this paper, Allen and Tett took this point further by advocating two independent estimates of C, one for estimation and the other for model validation.)

- **Correlation-Based Detection.** There is an analogy with methods based solely on correlations, especially when $p = 1$.

- **Statistical v. Practical Significance.** Possible that $\mathbf{a} \approx \mathbf{0}$ but H_0 still rejected.

Bayesian Climate Change Assessment

Levine, Berliner and Shea (*J. Clim* **13**, 3805–3820, 2000)

Idea: Present and alternative Bayesian viewpoint of detection and attribution procedures

Introduction

Fundamental model of structure

$$\text{Observations} \;=\; \mathbf{g} \times a + \;\text{noise.}$$

- 1961–1998 observational data (East Anglia) on $5^{\text{o}} \times 5^{\text{o}}$ grid
- Climate model output (CSM) interpolated to same grid
  - 300-year control run for estimating natural climate variability
  - 120-year CO_2 forced run in combination with control run to construct CO_2 fingerprint, written $\mathbf{g}$.

See Figure 1 for $\mathbf{g}$.

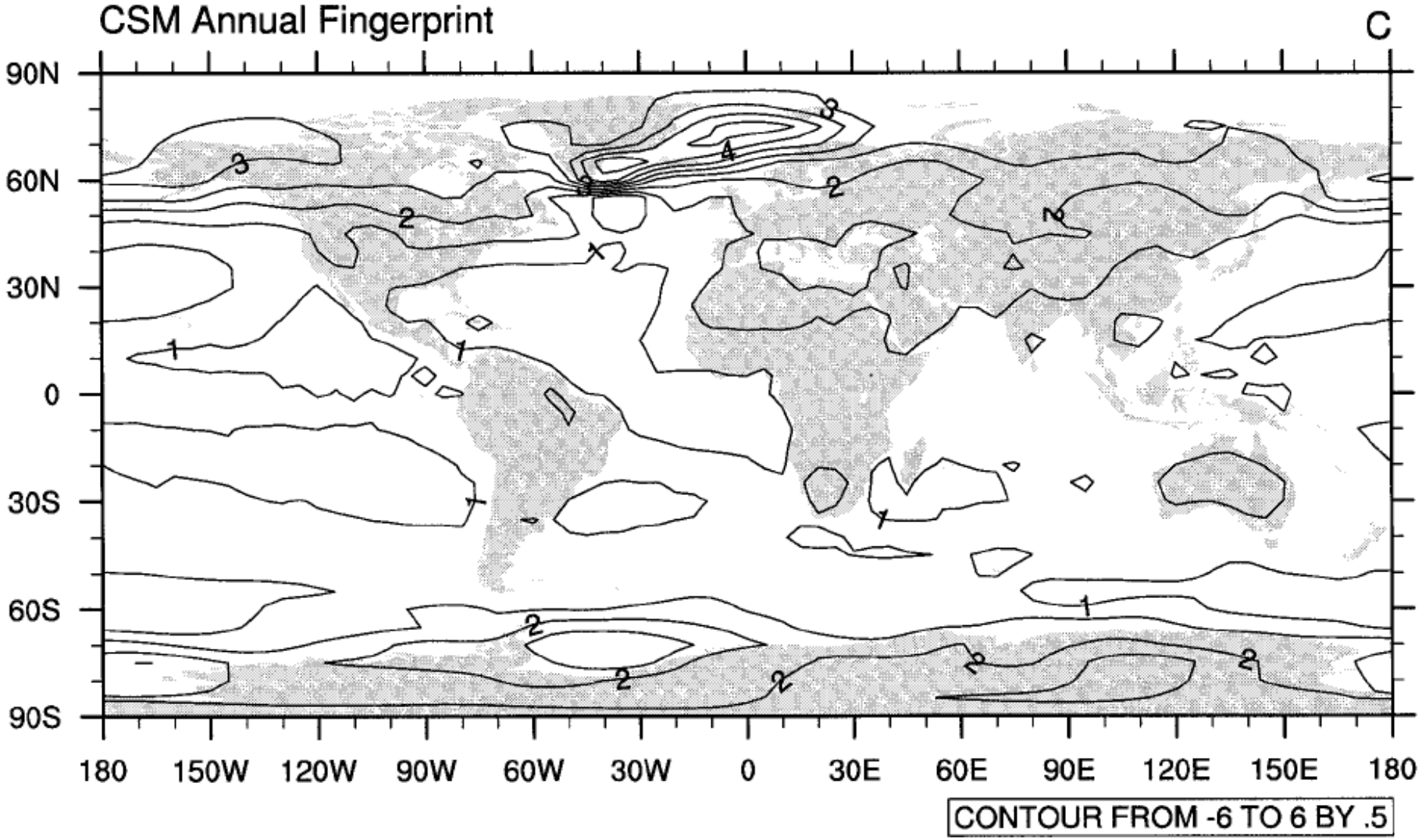

FIG. 1. Fingerprint pattern developed from NCAR CSM control and CO_2 runs on a 5° × 5° grid.

Reasons for adopting a Bayesian approach

- Inability to adopt conventional practices for a cause and effect analysis, e.g. randomized trials

- Need to account for uncertainties in observational data as well as in our knowledge of the how the climate system works (though the present discussion is incomplete, e.g. in its treatment of spatial and temporal correlations)

- Standard significance testing does not account for distinction between statistical and practical significance; this is more easily achieved within a Bayesian framework (this has been a theme of much of Jim Berger's work on Bayesian approaches to hypothesis testing)

Likelihood Function

- Observational data — yearly averages for 38 years, n =2592 spatial grid points, do not interpolate missing observations
- $\mathbf{T}_t$ is n-vector of "true" anomalies in year $t = 1, ..., m$; $\mathbf{T}$ is mn-vector formed by concatenating all $\mathbf{T}_t$.
- $\mathbf{Y}_t$ is n_t-vector of observed anomalies in year t ($n_t \leq n$ because of missingness); $\mathbf{Y}$ is concatenated vector.
- Write $\mathbf{Y}_t|\mathbf{T}_t, D_t \sim N[X_t T_t, D_t]$ where X_t is $n_t \times n$ incidence matrix. Similarly $\mathbf{Y}|\mathbf{T}, D \sim N[XT, D]$.
- Suppose $\mathbf{T_t}|a, \mathbf{g}, \mathbf{\Sigma} \sim N[\mathbf{g}a, \mathbf{\Sigma}]$.
- Unconditionally, $Y|a, \mathbf{g}, D, \mathbf{\Sigma} \sim N[X\mathbf{G}a, D + X\mathbf{\Sigma}X^T]$ ($\mathbf{G}$ is vector of m replicates of $\mathbf{g}$; $\mathbf{\Sigma}$ is space-time covariance matrix, assumed separable)

Some simplifications

Do not attempt to model uncertainties in D, $\mathbf{\Sigma}$ or $\mathbf{g}$: concentrate on a.

Standard GLS estimator

$$\widehat{a}(\mathbf{Y}) \;=\; \frac{\mathbf{H}^T V^{-1}\mathbf{Y}}{\mathbf{H}^T V^{-1}\mathbf{H}}$$

where $\mathbf{H} = X\mathbf{G}$ and $V = D + X^T\mathbf{\Sigma}X$; distribution of $\widehat{a}(\mathbf{Y})$ is

$$\widehat{a}(\mathbf{Y}) \;\sim\; N[a, \sigma^2]$$

where $\sigma^2 = (H^T V^{-1} H)^{-1}$.

Prior distribution for $\mathbf{a}$.

Assume a mixture prior:

$$\pi(a) \sim pN[0, \tau^2] + (1-p)N[\mu_A, \tau_A^2]$$

Estimates of τ^2, τ_A^2, μ_A derived from climate model runs under both control and forced conditions.

Posterior distribution for $\mathbf{a}$.

Mixture prior leads to mixture posterior

$$\begin{aligned}
\pi(a|\hat{a}) &= p(\hat{a})N[\mu(a|\hat{a}), \tau^2(a|\hat{a})] + \{1 - p(\hat{a})\}N[\mu_A(a|\hat{a}), \tau_A^2(a|\hat{a})], \\
\mu(a|\hat{a}) &= \frac{\tau^2}{\tau^2 + \sigma^2}\hat{a}, \\
\tau^2(a|\hat{a}) &= \frac{\tau^2\sigma^2}{\tau^2 + \sigma^2}, \\
\mu_A(a|\hat{a}) &= \frac{\tau_A^2}{\tau_A^2 + \sigma^2}\hat{a} + \frac{\sigma^2}{\tau_A^2 + \sigma^2}\mu_A, \\
\tau^2(a|\hat{a}) &= \frac{\tau_A^2\sigma^2}{\tau_A^2 + \sigma^2},
\end{aligned}$$

and $p(\hat{a})$ given by

$$\left[1 + \left(\frac{1-p}{p}\right)\sqrt{\frac{\tau^2 + \sigma^2}{\tau_A^2 + \sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{(\hat{a} - \mu_A)^2}{\tau_A^2 + \sigma^2} - \frac{\hat{a}^2}{\tau^2 + \sigma^2}\right)\right\}\right]^{-1}.$$

"Detection and Attribution"

Define "detection set" $\mathcal{D}$ (around $a = 0$) and "attribution set" $\mathcal{A}$ (around $a = \mu_A$) to represent subsets of physical interest, then

- "Detection" is said to occur if $\Pr\{a \in \mathcal{D}|\hat{a}\}$ is small,
- "Attribution" is said to occur if both (i) $\Pr\{a \in \mathcal{A}|\hat{a}\}$ is large, and (ii) $1 - p(\hat{a})$ is large.

"Robust Bayesian" viewpoint

Some prior parameters uncertain, especially $\mu_A,\ \tau_A$, so consider families of priors

$$\Gamma \ = \ \{\pi(a):\ \mu_\ell \le \mu_A \le \mu_u;\ \tau_\ell \le \tau_A \le \tau_u\}\,.$$

It's possible to calculate the maximum and minimum of $p(\widehat{a})$ over these subsets.

Results

Considered four subsets of time period, (a) 1961–1998, (b) 1970–1998, (c) 1980–1998, (d) 1988–1998.

Here used prior $p = 0.5$.

Fig. 2: Prior, likelihood and posterior distribution for a, for each of four time periods.

Table 1: Numerical summaries of posterior distribution

Conclusion: As $p(a|\widehat{a})$ moves closer to 0, stronger evidence that climate model is correct.

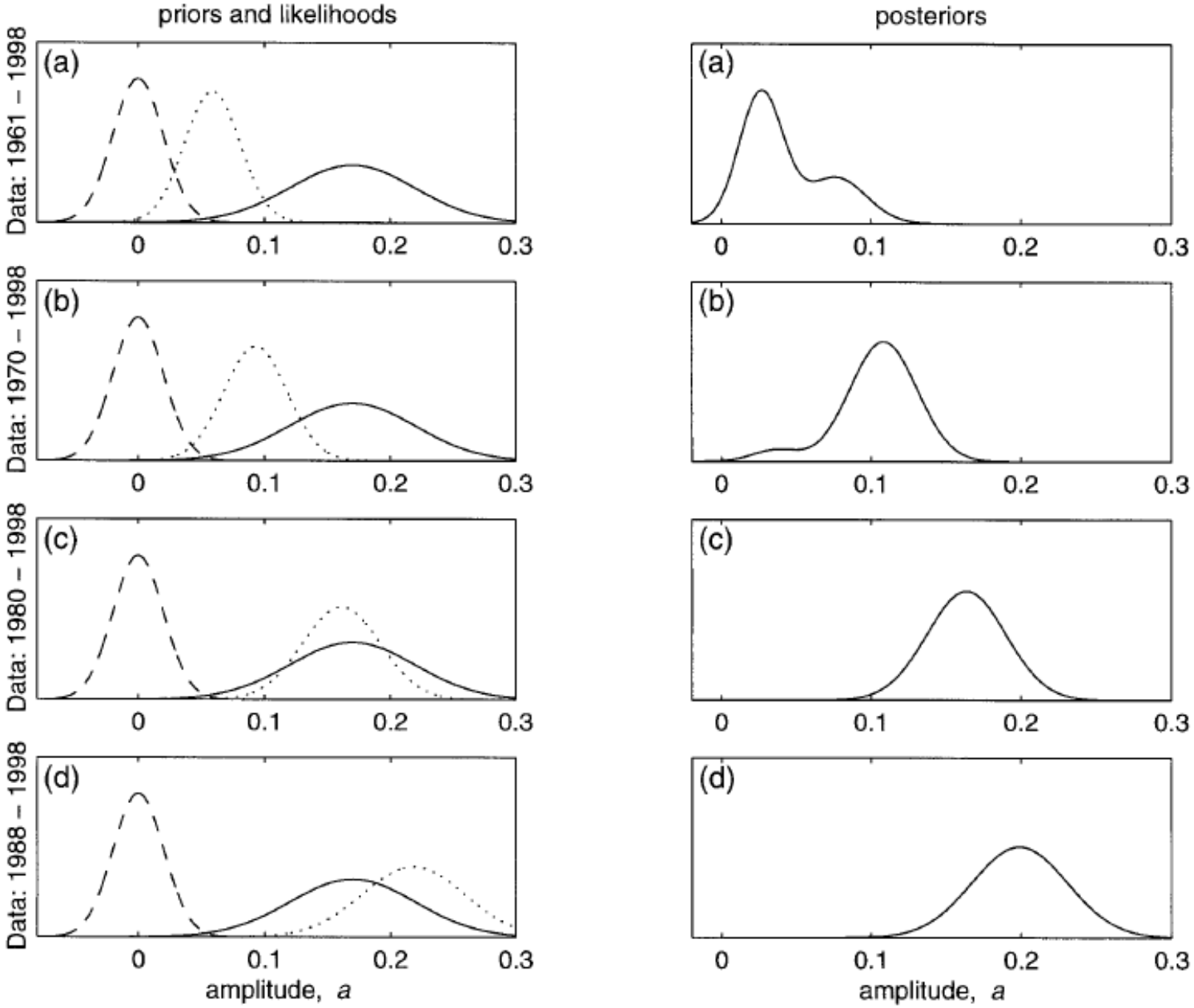

FIG. 2. Likelihood function, prior distribution, and posterior distribution of a using our NCAR CSM fingerprint. For each of the time periods (a)–(d): (left) the likelihood (dotted line) and prior distribution components [anthropogenic CO_2 forcing (solid line); no anthropogenic impacts (dashed line)], and (right) the posterior mixture distribution (11).

TABLE 1. For each of the four time periods a–d: GLS estimate $\hat{a}$ of a and associated standard deviation σ; mean and associated standard deviation of a under the first component (no climate change) of the posterior mixture in (11); posterior mean and associated standard deviation under the second component (CO_2 forcing) of the posterior mixture in (11); posterior weight , $p(\hat{a})$, assuming prior weight $p = 0.5$, $\tau = 0.02$, $\mu_A = 0.17$, and $\tau_A = 0.05$.

Time period	1961–98		1970–98		1980–98		1988–98	
$\hat{a}$, σ	0.059	0.022	0.093	0.025	0.160	0.031	0.220	0.041
$\mu(a \mid \hat{a})$, $\tau(a \mid \hat{a})$	0.027	0.015	0.036	0.016	0.047	0.017	0.042	0.018
$\mu_A(a \mid \hat{a})$, $\tau_A(a \mid \hat{a})$	0.077	0.020	0.108	0.022	0.164	0.026	0.200	0.032
$p(\hat{a})$		0.671		0.062		0.0001		0.00001

Fig 3: Posterior probability of $p(\widehat{a})$ versus prior probability p

Figs 4 and 5: Detection and attribution results based on $\mathcal{D} = (-0.05, 0.05)$, $\mathcal{A} = (\mu_A - 0.05, \mu_A + 0.05)$.

Table 2: Robust Bayesian results

Evem for time period (b), results are much stronger in favor of both detection and attribution than for time period (a). Results become overwhelming for time periods (c) and (d).

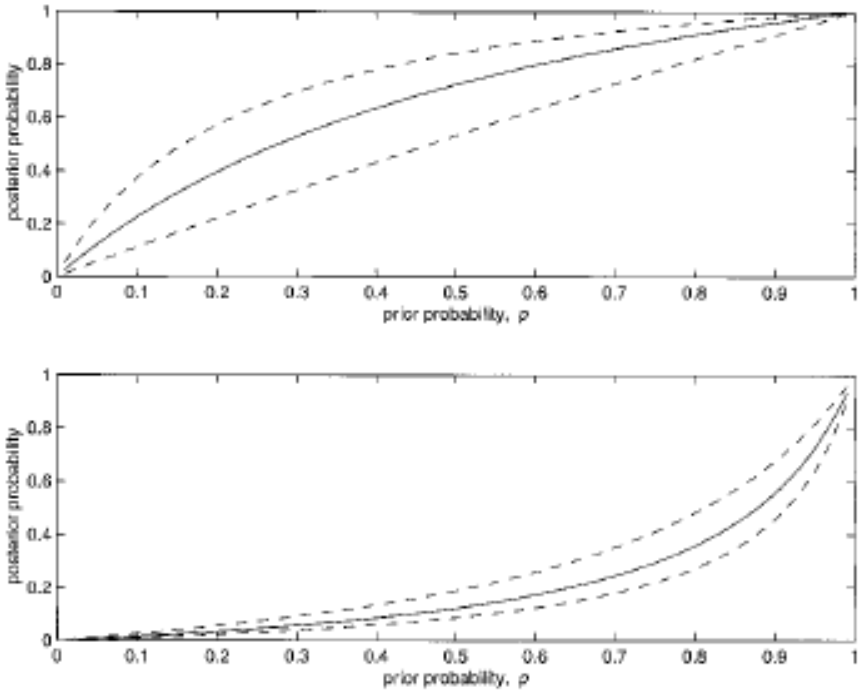

FIG. 3. Posterior mixture probability $p(\hat{d})$ versus the prior mixture probability p based on two data subsets: (top) 1961–98 and (bottom) 1970–98. The solid line in each denotes the posterior probability computed from the data. The dotted lines denote the upper and lower bounds on the posterior probabilities from the robust Bayesian analysis.

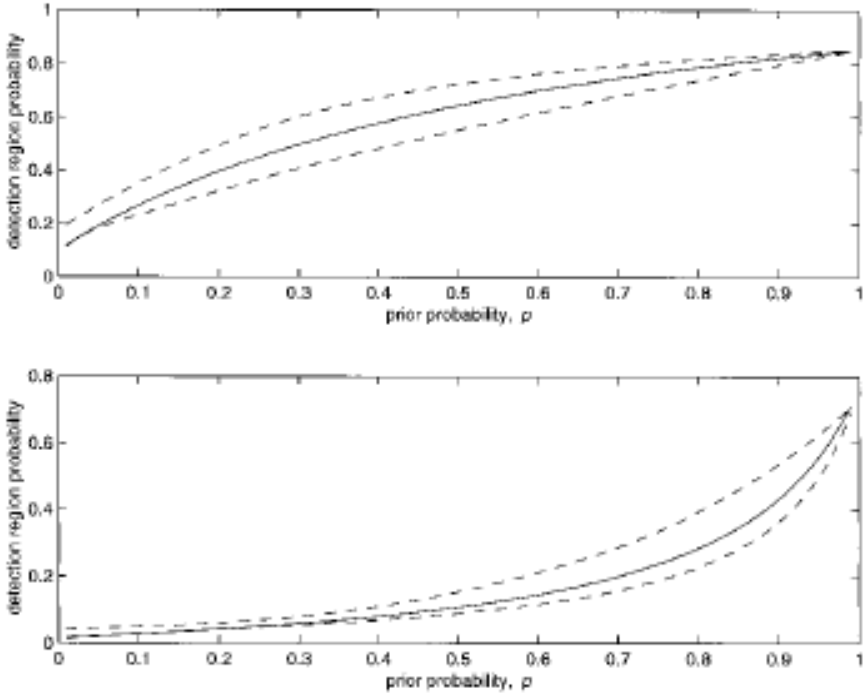

FIG. 4. Posterior probability $\Pr(a \in [-0.05, 0.05] | \hat{d})$ as a function of the prior probability p based on two time periods: (top) 1961–98 and (bottom) 1970–98. The solid lines in each denotes the posterior probabilities for the primary prior. The dotted lines give the upper and lower bounds on these probabilities from the robust Bayesian analysis.

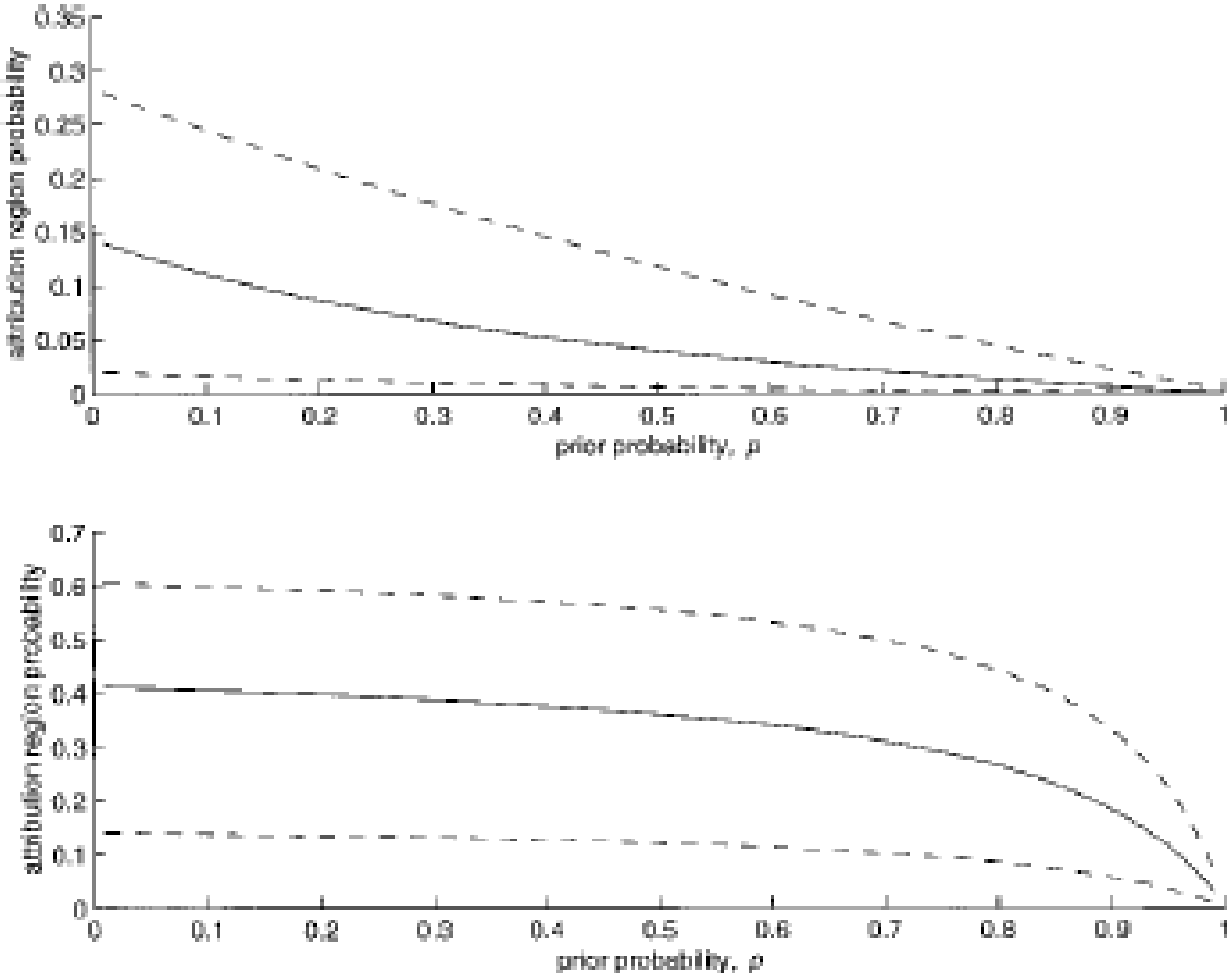

FIG. 5. Posterior probability $\Pr(a \in [\mu_A - 0.05, \mu_A + 0.05] \mid \hat{d})$ as a function of the prior probability p based on two time periods: (top) 1961–98 and (bottom) 1970–98. The solid lines in each denotes the posterior probabilities for the data. The dotted lines give the upper and lower bounds on these probabilities from the robust Bayesian analysis.

TABLE 2. Upper and lower bounds on data-based contributions to the odds ratio in favor of the CO_2 forced posterior model for each of the four data subsets.

1961–98		1970–98		1980–98		1988–98	
m	*M*	*m*	*M*	*m*	*M*	*m*	*M*
1.3339	0.2174	23.1546	8.6383	8776	6283.24	71 150	43 524

Results based on alternative fingerprint from paper of Santer *et al* (1995)

Fig 6: Graphic of $\mathbf{g}$ in this case

Table 3: Compare $\widehat{a}$ for two models (those for Santer model consistently smaller, but this is not surprising when comparing two different vectors $\mathbf{g}$)

Figs. 7–8: Posterior probabilities (less definitive for whole time period, but the Santer fingerprint leads to very strong attribution results in time period (d))

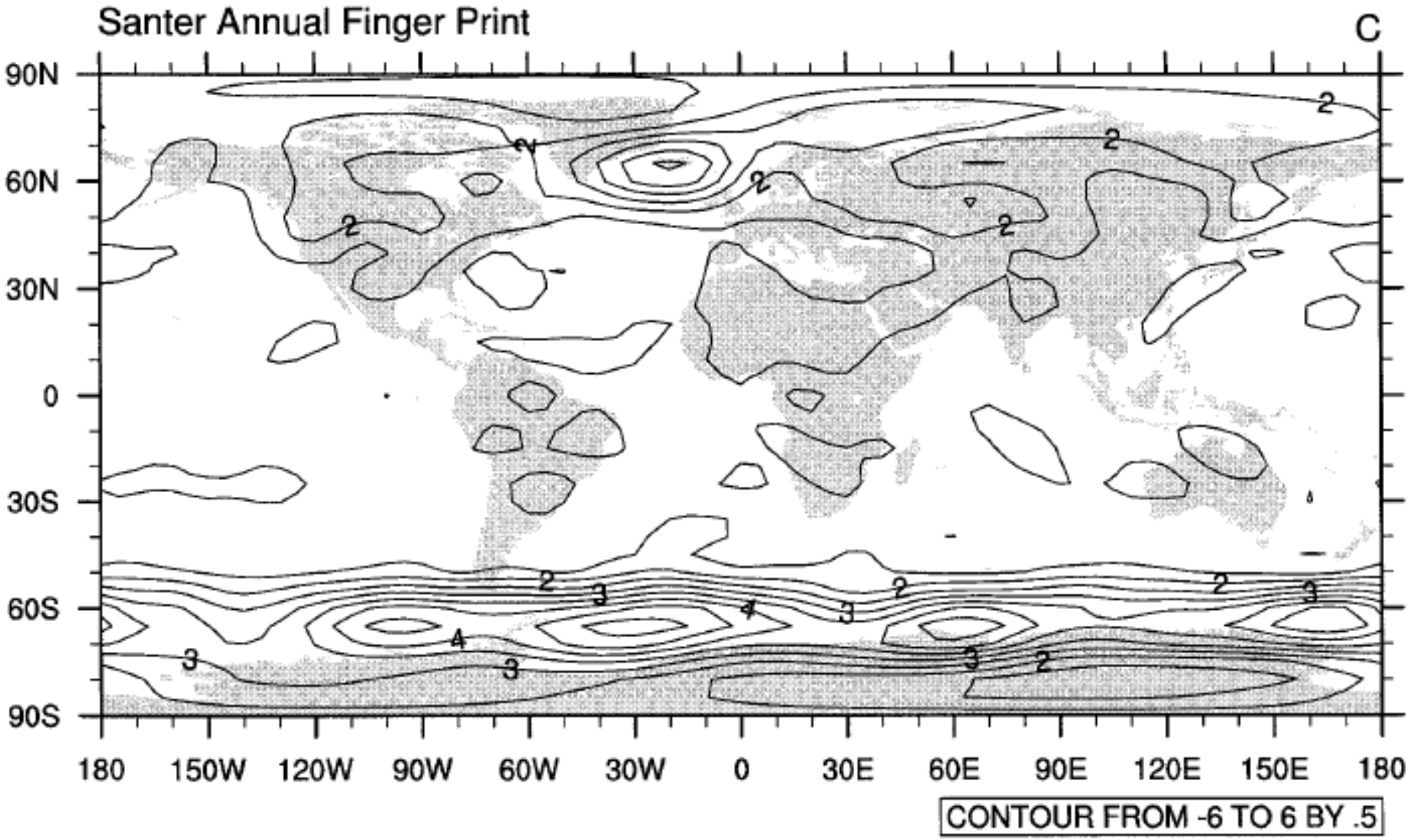

FIG. 6. Fingerprint pattern generated by Santer et al. (1995), after their Fig. 3b, on a 5° × 5° grid.

TABLE 3. GLS estimates of a and associated standard deviations for each of the four data subsets using the Santer et al. (1995) fingerprint.

	1961–98		1970–98		1980–98		1988–98	
Fingerprint	$\hat{a}$	σ	$\hat{a}$	σ	$\hat{a}$	σ	$\hat{a}$	σ
Santer	0.0394	0.017	0.062	0.019	0.1117	0.024	0.1508	0.031
CSM	0.0589	0.022	0.093	0.025	0.1612	0.031	0.2200	0.041

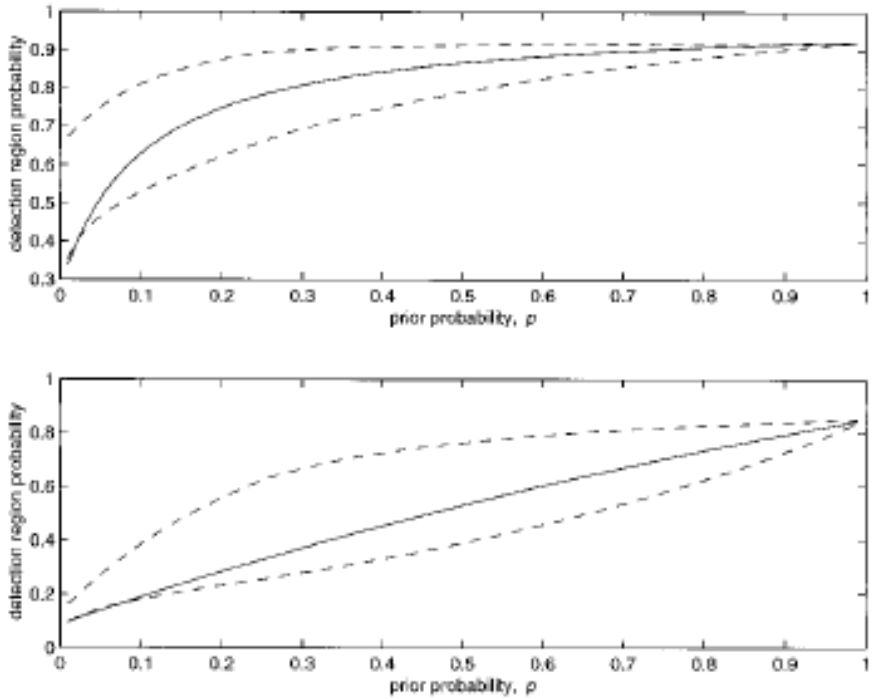

FIG. 7. Posterior probability $\Pr(a \in [-0.05, 0.05] | \hat{a})$ as a function of the prior probability p based on two time periods: (top) 1961–98 and (bottom) 1970–98. The solid lines in each denotes the posterior probabilities for the data. The dotted lines give the upper and lower bounds on these probabilities from the robust Bayesian analysis.

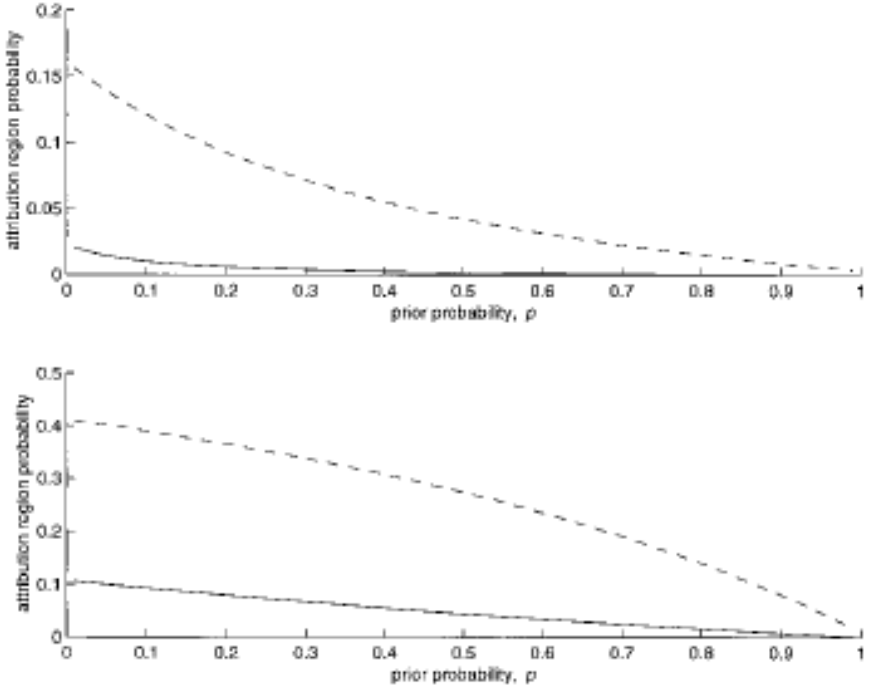

FIG. 8. Posterior probability $\Pr(a \in [\mu_A - 0.05, \mu_A + 0.05] | \hat{a})$ as a function of the prior probability p based on two time periods: (top) 1961–98 and (bottom) 1970–98. The solid lines in each denotes the posterior probabilities for the data. The dotted lines give the upper and lower bounds on these probabilities from the robust Bayesian analysis.

Comparison with traditional results

See Table 4

Traditional results more likely to reject H_0 when $\hat{a}$ very close to 0 (so Bayesian procedure more conservative for detection)

But situation "somewhat reversed" for attribution.

TABLE 4. Significance probabilities for traditional, non-Bayesian detection and attribution test results. Detection is suggested by very small values; attribution may be suggested by moderate or large values.

	1961–98		1970–98		1980–98		1988–98	
	CSM	Santer	CSM	Santer	CSM	Santer	CSM	Santer
Detection	0.004	0.010	0.000	0.001	0.000	0.000	0.000	0.000
Attribution	0.000	0.000	0.000	0.000	0.744	0.037	0.220	0.720

Superensemble Based Climate Forecasting

Berliner and Kim (*J. Clim* **21**, 1891–1910, 2008)

- Develop statistical data models to combine ensembles from multiple climate models in a fashion that accounts for uncertainty

- "Superensemble experimental design"

  "Superensembles" means *ensembles from multiple climate models*

- Bayesian climate modeling and forecasting

- Forecasting of future hemispheric monthly-averaged surface temperatures under different SRES scenarios

Outline of approach (univariate)

Suppose we have scalar X, and let $\mathbf{Y}_m$ be a vector of n_m model runs for the same variable from model m, $m = 1, ..., M$.

Construct distribution of $\mathbf{Y}_m$ given X as follows:

- Given X, generate $\beta|X \sim N[X, \sigma^2_\beta]$.
- Also generate "model biases" $b_1, ..., b_M$ such that $b_m \sim N[\mu_m, \sigma^2_{b_m}]$ (independent for different m, though this assumption could be relaxed)
- $\mathbf{Y}_m|\beta, b_m \sim N[(\beta + b_m)\mathbf{1}_m, \sigma^2_{Y_m} I_{n_m \times n_m}]$
- Hence combine to generate conditional density $[\mathbf{Y}|X]$ in terms of parameters $\sigma^2_\beta, \mu_m, \sigma^2_{b_m}, \sigma^2_{Y_m}, m = 1, ..., M$ (prior distributions on all of those)
- *Also* have a model for evolution of X in time (e.g. autoregressive time series model)

Multiple climate variables

Extend the univariate model using Kronecker products, e.g.

$$\begin{aligned} \boldsymbol{\beta}|\mathbf{X} &\sim N[\mathbf{X}, \boldsymbol{\Sigma}_{\boldsymbol{\beta}}], \\ \mathbf{b}_m &\sim N[\boldsymbol{\mu}_m, \boldsymbol{\Sigma}_{\mathbf{b}_m}], \\ vec(\mathbf{Y}_m)|\boldsymbol{\beta}, \mathbf{b}_m &\sim N[\mathbf{1}_{n_m} \otimes (\boldsymbol{\beta} + \mathbf{b}_m), I_{n_m \times n_m} \otimes \boldsymbol{\Sigma}_{\mathbf{Y}_m}]. \end{aligned}$$

Outline of Bayesian strategy:

Use superscripts p and f to denote "present" and "future".

We have models for

- $[\mathbf{X}^f \mid \mathbf{X}^p]$ (AR assumption)
- $[\mathbf{Y}^p \mid \mathbf{X}^p]$
- $[\mathbf{Y}^f \mid \mathbf{X}^f]$
- Assume conditionally independent
- Hence deduce joint $[\mathbf{X}^f, \mathbf{Y}^p, \mathbf{Y}^f \mid \mathbf{X}^p]$
- Hence $[\mathbf{X}^f \mid \mathbf{X}^p, \mathbf{Y}^p, \mathbf{Y}^f]$ by Bayes theorem

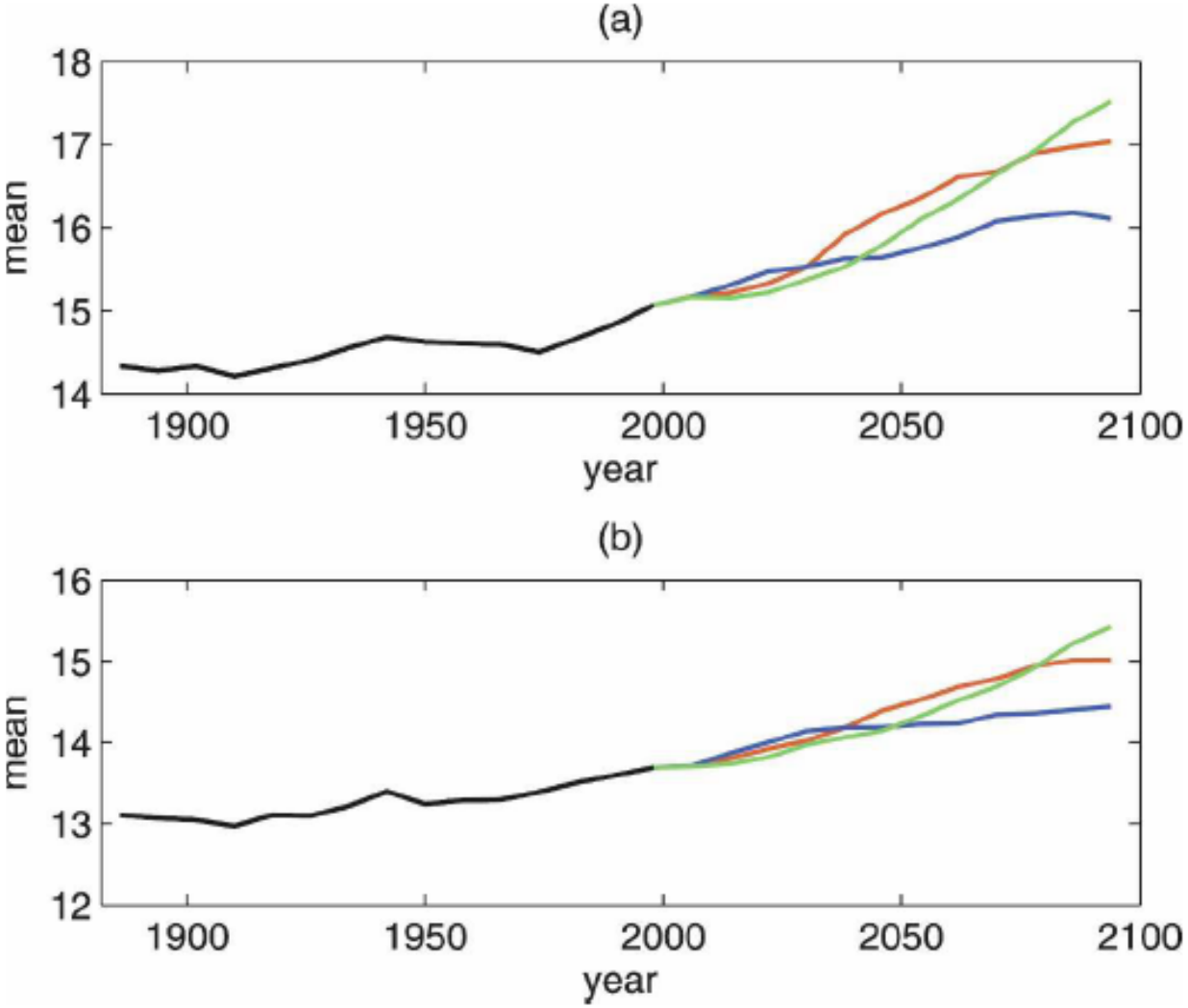

FIG. 1. Hemispheric process model means. (a) NH and (b) SH process model mean parameters (denoted by α's in text). One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.

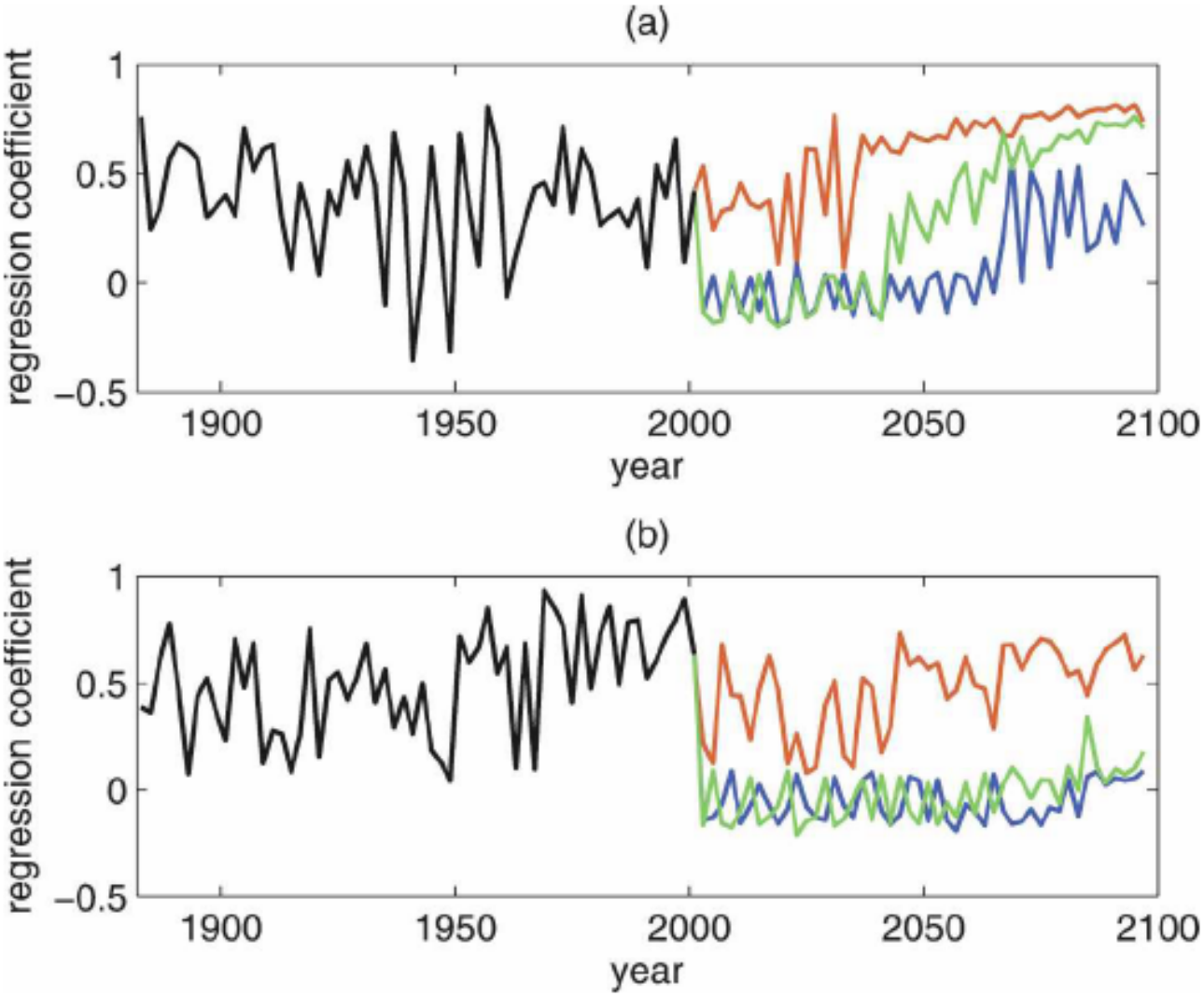

FIG. 2. Hemispheric process autoregression parameters. (a) NH and (b) SH process model autoregression coefficients (denoted by η's in text). One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.

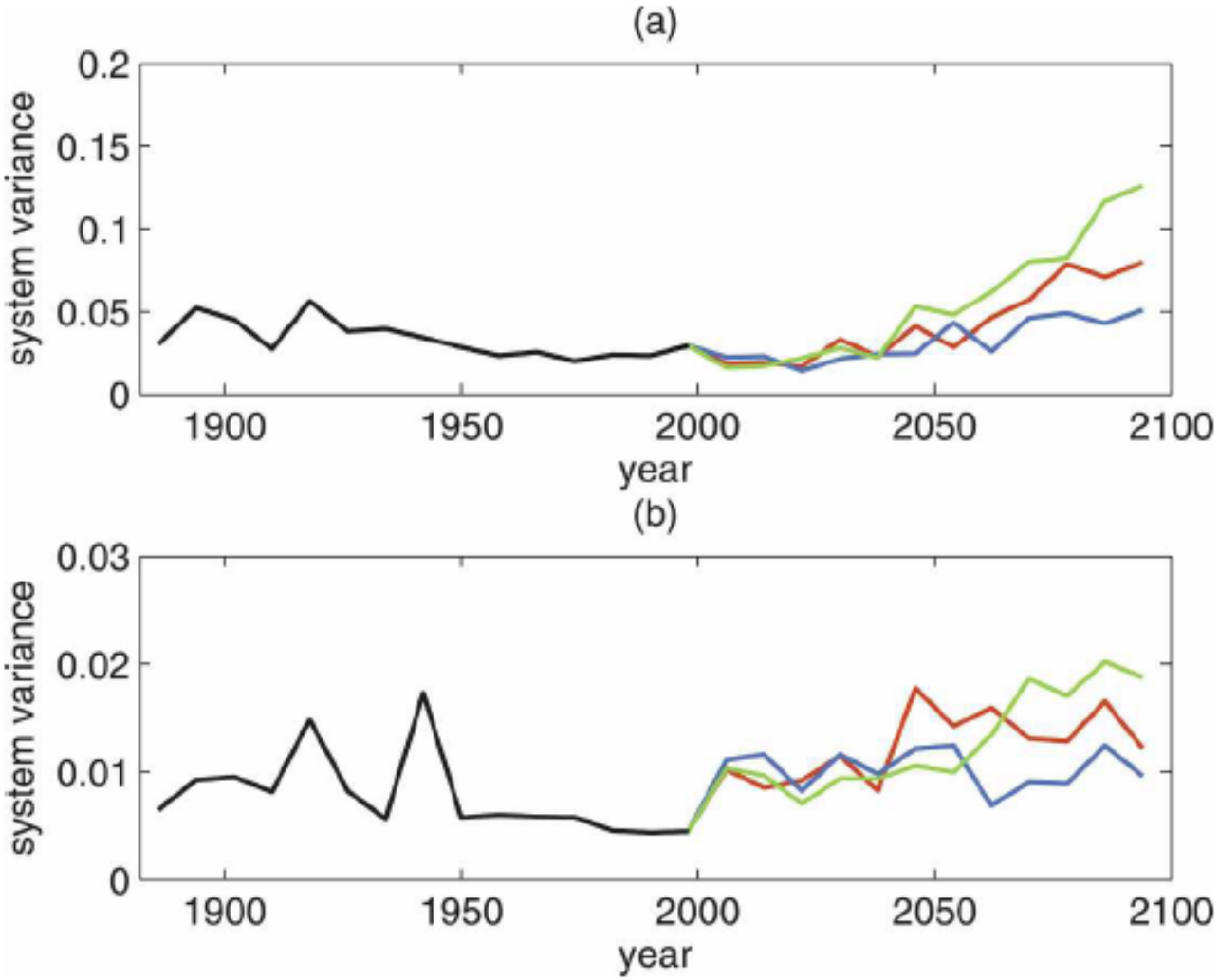

FIG. 3. Hemispheric process variances. (a) NH and (b) SH process variances (denoted by σ_i^2's in text). One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.

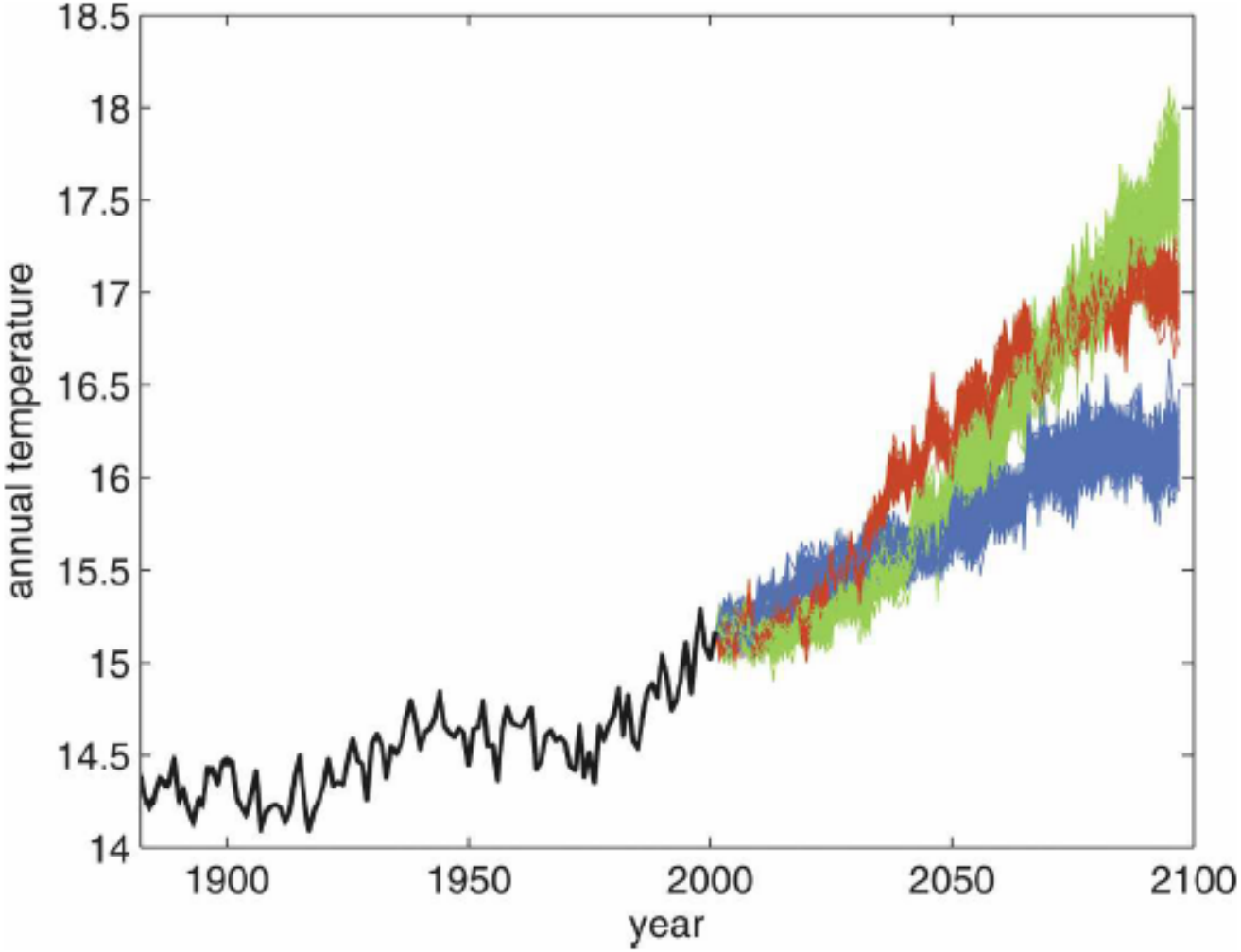

FIG. 4. NH temperature reconstructions and forecasts. One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black; observations are also plotted here) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.

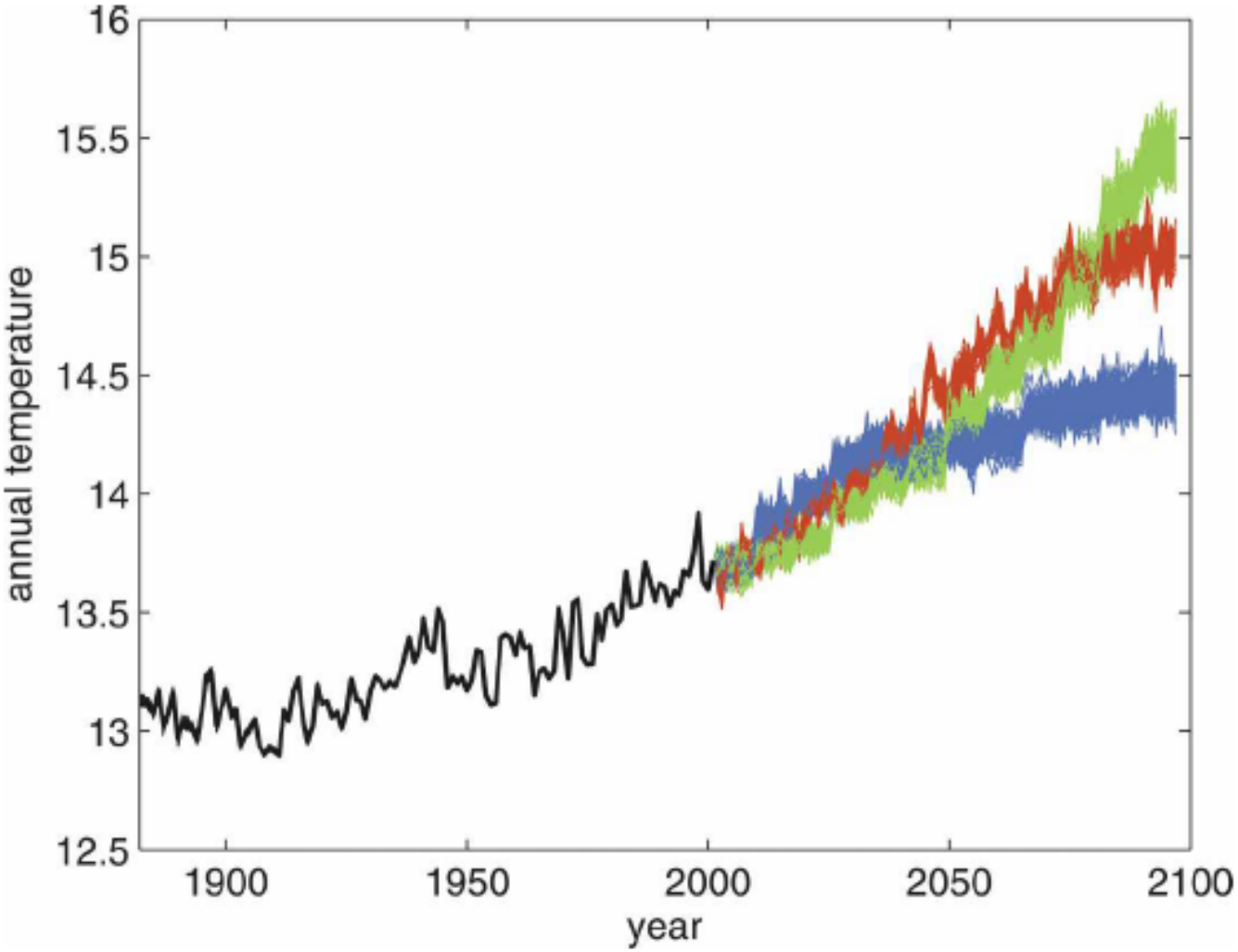

FIG. 5. SH temperature reconstructions and forecasts. One hundred ensembles generated from the posterior distribution are plotted in each of the cases: observation period 1882–2001 (black) (observations are also plotted here) and the forecast period 2002–97 for SRES scenarios A1B (red), B1 (blue), and A2 (green), respectively.

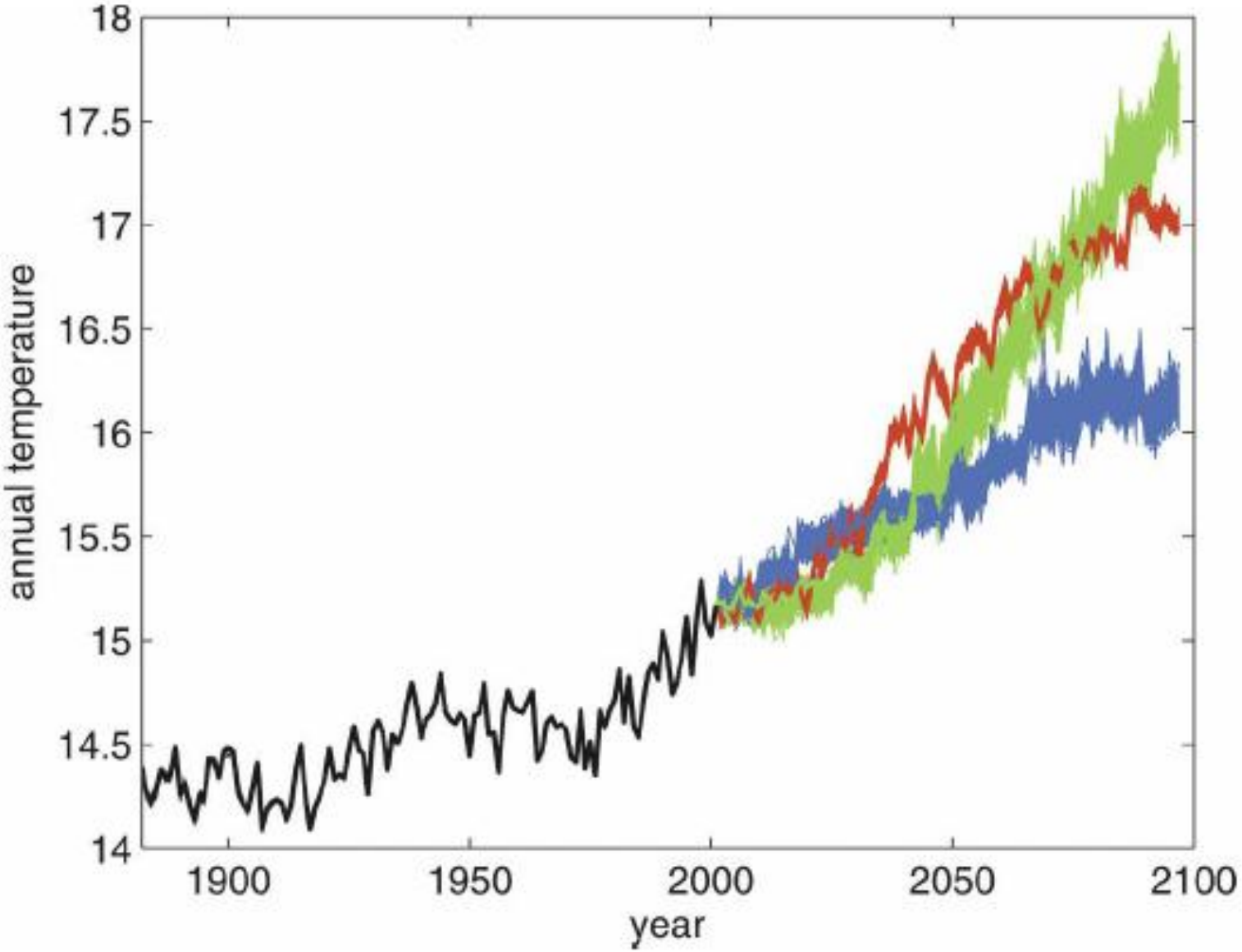

FIG. 6. NH temperature reconstructions. Same as in Fig. 4 but based on a model assuming constant biases for all times in the forecast period.

TABLE 1. Means (std dev) of intercept and slope of regression of means on CO_2 based on 5000 realizations from the posterior distribution.

	Intercept		Slope	
	NH	SH	NH	SH
Observation	10.256 (0.103)	9.922 (0.078)	0.014 (3.4×10^{-04})	0.011 (2.6×10^{-04})
Forecast				
A1B	12.383 (0.061)	11.842 (0.047)	6.76×10^{-03} (1.3×10^{-04})	4.59×10^{-03} (8.8×10^{-05})
B1	12.631 (0.104)	12.199 (0.105)	5.78×10^{-03} (2.1×10^{-04})	3.72×10^{-03} (2.2×10^{-04})
A2	12.360 (0.109)	11.724 (0.040)	6.40×10^{-03} (1.9×10^{-04})	4.41×10^{-03} (7.2×10^{-05})

TABLE 2. Means (std dev) of biases $\mathbf{b}^1$ and $\mathbf{b}^2$ based on 5000 realizations from the posterior distribution.

Scenario		Period	PCM	CCSM
A1B	NH	2002–31	2.1777 (0.2481)	−0.1635 (0.3326)
		2032–51	2.9447 (0.3154)	−0.1043 (0.4512)
		2052–71	2.6964 (0.3556)	−0.1406 (0.4723)
		2072–91	2.7923 (0.3431)	−0.1228 (0.4615)
		2092–97	2.5123 (0.4832)	−0.3215 (0.5101)
	SH	2002–31	0.6559 (0.1031)	−0.0771 (0.0234)
		2032–51	0.3204 (0.1262)	0.3150 (0.0631)
		2052–71	0.2550 (0.1416)	0.4121 (0.0843)
		2072–91	0.2977 (0.1323)	0.4575 (0.0712)
		2092–97	0.3643 (0.1853)	0.4352 (0.1323)
B1	NH	2002–31	2.1707 (0.1856)	−0.1186 (0.3812)
		2032–51	2.8037 (0.2015)	−0.0122 (0.4262)
		2052–71	2.7395 (0.2963)	−0.1874 (0.4732)
		2072–91	2.8523 (0.2831)	−0.2550 (0.5213)
		2092–97	2.5412 (0.3612)	−0.2743 (0.6722)
	SH	2002–31	0.6379 (0.0812)	−0.0239 (0.2502)
		2032–51	0.4743 (0.1023)	0.2016 (0.3107)
		2052–71	0.2505 (0.1274)	0.2929 (0.3212)
		2072–91	0.3403 (0.1152)	0.3307 (0.3551)
		2092–97	0.4123 (0.1891)	0.2133 (0.4214)
A2	NH	2002–31	2.1001 (0.1525)	No data
		2032–51	2.7188 (0.1712)	
		2052–71	2.6367 (0.1842)	
		2072–91	2.5975 (0.1444)	
		2092–97	2.3245 (0.3132)	
	SH	2002–31	0.5132 (0.0523)	No data
		2031–51	0.1719 (0.0912)	
		2052–71	0.1023 (0.1142)	
		2072–91	0.2246 (0.1073)	
		2092–97	0.2812 (0.1412)	

TABLE 3. Means (std dev) of model error covariance matrices based on 5000 realizations from the posterior distribution, where Σ_{Y_1} is for PCM and Σ_{Y_2} is for CCSM.

Scenario		NH variance	SH variance	Covariance
A1B	Σ_{Y_1}	0.144 (0.003)	0.068 (0.002)	0.012 (0.002)
	Σ_{Y_2}	0.338 (0.015)	0.206 (0.010)	0.101 (0.010)
	Σ_{β}	0.045 (0.002)	0.032 (0.002)	0.001 (0.002)
B1	Σ_{Y_1}	0.174 (0.005)	0.089 (0.003)	0.014 (0.002)
	Σ_{Y_2}	0.326 (0.016)	0.153 (0.007)	0.022 (0.006)
	Σ_{β}	0.085 (0.004)	0.048 (0.002)	0.001 (0.002)
A2	Σ_{Y_1}	0.167 (0.003)	0.085 (0.002)	0.012 (0.003)
	Σ_{β}	0.095 (0.007)	0.055 (0.003)	−0.001 (0.003)

Detection and Attribution Analysis for Precipitation

Reference: Zhang *et al.*, **Nature** 2007 (see earlier detection and attribution notes)

Datasets

Latitude-band averages

- 75 years
- 11 ten-degree latitude bands (from 40^{o}S to 70^{o}N)
- One observational dataset
- Several forced model runs under different forcings (ANT, ALL, NAT)
- Control runs — typically 6 control runs of 1000 years broken up into 75-year segments

Traditional D&A Analysis

1. For each of 11 latitude bands, linear trend is calculated by OLS, including control runs

2. Covariance matrix from control run used to define the weights for EOF decomposition

3. Truncate the EOFs, e.g. by Allen-Tett test (anticipate ≈ 5 EOFs retained

4. Same formation of linear trends and EOFs applied to forced model runs to develop a set of comparison vectors.

5. Optimal weights for D&A defined by regressing observational data EOFs on forcing model EOFs. Calculate standard errors, tests of significance, etc.

Estimation of the Covariance Matrix

The data we have available consists of 17 runs of 75 years' data from control runs of HADCM3. The data are 11-dimensional because of the 11 latitude bands. The covariance matrix C is defined to be the 11×11 covariance matrix of estimators of linear trends over the 75 years.

Possible methods of estimating C include:

Method 1

Calculate linear trends for each of the 11 latitude bands and each of the 17 runs; then estimate C (say, as $\hat{C}_1$) as the sample covariance matrix

$$\hat{C}_1 = \frac{(B - \bar{B})^T (B - \bar{B})}{16} \tag{1}$$

where B is the 17×11 matrix of linear trend estimates and $\bar{B}$ is the mean of B over all rows.

Method 2

Suppose Y_{ijk} is observation in year i from latitude band j and control model run k. Suppose that, for each i and k, $\{Y_{ijk},\ j = 1, ..., 11\}$ is an independent vector from $N[0, V]$, where V is the 11×11 covariance matrix associated with a single year of control data. Linear trend estimator for region j and model run k given by

$$B_{jk} \ = \ \sum_{i=1}^{75} w_i Y_{ijk} \tag{2}$$

for weights w_i. Then for any two regions $j,\ j'$,

$$\text{Cov}(B_{jk}, B_{j'k}) \ = \ \sum_{i=1}^{75} w_i^2 V_{jj'}. \tag{3}$$

If $\widehat{V}$ is sample covariance matrix, estimate

$$\widehat{C}_2 \ = \ (\sum w_i^2)\widehat{V}. \tag{4}$$

Method 3

Correct for autocorrelation in method 2.

Assume h-order autocovariances negligible for $|h| > H$ some given H. Then we can modify (3) to

$$\text{Cov}(B_{jk}, B_{j'k}) = \sum_{h=-H}^{H} \sum_{i=1}^{75} w_i w_{i+h} \text{Cov}(Y_{ijk}, Y_{(i+h)j'k}) \quad (5)$$

where w_{i+h} is defined to be 0 if $i + h < 1$ or $i + h > 75$.

If $\widehat{v}_{hjj'}$ denotes sample covariance of Y_{ijk} and $Y_{(i+h)j'k}$ (computed from all 17×75 control years of data) then this implies a covariance estimator $\widehat{C}_3$ with entries

$$\widehat{C}_3(j, j') = \sum_{h=-H}^{H} \sum_{i=1}^{75} w_i w_{i+h} \widehat{v}_{hjj'}. \quad (6)$$

Looking for autocorrelations in the control runs

Sample autocorrelations and partial autocorrelations were calculated from the control data. They were considered statistically significant if outside the range $\pm\frac{2}{\sqrt{n}}$

These results suggest that significant autocorrelations do not persist beyond lag 2. However, as a test, the estimator $\hat{C}_3$ was computed for each of $H = 1, 2, 3, 4, 5$.

Testing one estimated covariance matrix against another

Advantage of $\hat{C}_1$ — makes no assumption at all about the autocorrelation structure, but it's based on a small sample (17).

$\hat{C}_2$ or $\hat{C}_3$ — far larger sample size, but maybe incorrect assumptions. Suggests a hypothesis test that of either $H_0: \ C = \hat{C}_2$ or $H_0: \ C = \hat{C}_3$, treating $\hat{C}_1$ as a sample covariance matrix.

Korin (1968) proposed likelihood ratio test statistic

$$L \ = \ N \left\{ \log \left(\frac{|\Sigma_0|}{|S|} \right) - p + \mathrm{tr}(S\Sigma_0^{-1}) \right\} \qquad (7)$$

where S is a $p \times p$ sample covariance matrix with N degrees of freedom, and Σ_0 is the null hypothesis value of the true covariance matrix. Sampling distributions by Bartlett correction to the χ^2 test, an alternative approximation proposed by Korin, or simulation.

Application

$\hat{C}_1$ against $\hat{C}_2$ (as null hypothesis): reject under Bartlett's approximation, accept under Korin's or simulation (p-values .047, .077, .11) but maybe not a clear-cut result.

However for $\hat{C}_1$ against $\hat{C}_3$ (as null hypothesis), with any of $H = 1, 2, 3, 4, 5$, accept null hypothesis. As a specific example, for $H = 2$, Bartlett's, Korin's and the simulation test yield p-values 0.33, 0.42 and 0.47.

Conclusion: $\hat{C}_3$ seems fine, and the actual value of H doesn't matter.

On the other hand, where we see real differences among the estimators is in computing their inverses. As an example, the diagonal entries of $\widehat{C}_1^{-1}$ are:

```
 202.1  50.5   8.1   7.5   4.9  44.4  18.4  55.3 124.0 338.0 136.3
```

The corresponding values for $\widehat{C}_3^{-1}$ with $H = 0 - 5$ are:

```
H=0: 29.9 20.6  5.1  4.9  5.5 11.1 11.7 27.4 50.5 29.7 24.5
H=1: 27.8 19.5  4.6  4.5  5.1 10.2 11.7 28.6 44.2 27.3 22.1
H=2: 28.7 20.6  4.8  4.6  5.1  9.7 11.5 31.6 40.7 28.4 24.8
H=3: 28.5 19.0  4.7  4.5  5.2  9.3 11.0 33.4 39.5 29.1 26.1
H=4: 29.1 20.5  4.5  4.4  5.2  8.9 11.1 33.5 39.8 29.2 25.4
H=5: 30.7 20.1  4.4  4.4  4.6  8.8 11.4 31.3 40.9 28.6 25.1
```

Evidently, there is much more stability in any of the versions of $\widehat{C}_3^{-1}$ (raising the question that maybe no dimension reduction is actually needed??)

Results Applied to D&A

Analysis using full data matrix (11 PCs)

Analyses using $\hat{C}_1, \hat{C}_2, \hat{C}_3 (H = 2)$

Models	$\hat{\beta}$	SE	t statistic	P-value
ALL	-0.043	0.354	-0.121	0.903
ANT	0.695	0.318	2.183	0.029
NAT	-1.701	0.479	-3.550	0.000

Models	$\hat{\beta}$	SE	t statistic	P-value
ALL	1.272	0.584	2.179	0.029
ANT	3.495	0.519	6.730	0.000
NAT	-3.950	0.736	-5.365	0.000

Models	$\hat{\beta}$	SE	t statistic	P-value
ALL	1.730	0.612	2.825	0.005
ANT	3.470	0.532	6.523	0.000
NAT	-4.279	0.777	-5.507	0.000

Analysis using 5 PCs

Analyses using $\hat{C}_1, \hat{C}_2, \hat{C}_3 (H = 2)$

Models	$\hat{\beta}$	SE	t statistic	P-value
ALL	3.888	1.345	2.891	0.004
ANT	3.370	1.195	2.821	0.005
NAT	8.084	4.205	1.922	0.055

Models	$\hat{\beta}$	SE	t statistic	P-value
ALL	3.300	0.941	3.508	0.000
ANT	3.486	0.968	3.601	0.000
NAT	2.396	1.902	1.259	0.208

Models	$\hat{\beta}$	SE	t statistic	P-value
ALL	2.375	0.990	2.399	0.016
ANT	3.270	1.252	2.613	0.009
NAT	2.312	2.199	1.051	0.293

Conclusions (preliminary)

Using the full data matrix, there is a lot of discrepancy between the results based on $\hat{C}_1$ and either of $\hat{C}_2$ or $\hat{C}_3$, but the results comparing $\hat{C}_2$ and $\hat{C}_3$ are quite close

Using 5 PCs, there is much better agreement between the three covariance models, but results are quite different from those using all 11 PCs

Based on all 11 PCs with either $\hat{C}_2$ or $\hat{C}_3$, we may claim to have attribution for the all forcings model, but not for either ANT or NAT.