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STOR834 p.100

(c) Wadsworth, Tawn, Davison & Elton (JRSSB, 2016)

Starts by defining
$$\chi = \lim_{u \to 1} P\left[F_1(Z_1) > u \mid F_2(Z_2) > u\right]$$

$\chi > 0$: asy. dep

$\chi = 0$: asy. indep

(higher dimensions: multiple coefficients of similar structure)

Define $(X_P, Y_P) = \left[\{1-F_1(Z_1)\}^{-1}, (1-F_2(Z_2))^{-1}\right]$

Marginal standard Pareto

Define $W = \frac{(X_P, Y_P)}{\|(X_P, Y_P)\|_a}$, $R = \|(X_P, Y_P)\|_b$

$\|\cdot\|_a$, $\|\cdot\|_b$ arbitrary norms but maybe $a=b=1$

$$\left[\|(x,y)\|_a = (|x|^a + |y|^a)^{1/a} \quad ?? \right]$$

Then $\lim_{t \to \infty} P\{W \in B, R > t(r+1) \mid R > t\} = \frac{H(B)}{r+1}$

$r \ge 0$, $B \subseteq S^a = \{\underset{\sim}{w} \in \mathbb{R}_+^2, \|\underset{\sim}{w}\|_a = 1\}$

Idea: try other marginal distributions.

Assume common marginal F on $(0, x^F)$, $x^F \le \infty$

$$(X, Y) = F^{\leftarrow}\{F_1(Z_1)\},\ F^{\leftarrow}\{F_2(Z_2)\},$$

Assume for some norm $\|\cdot\|_*$ and $a(t)>0$, $b(t)\in\mathbb{R}$,

$$\cancel{\lim_{t\to\infty} P\left[\frac{X}{X+Y}\le w,\ \|(X,Y)\|_* > a(t)r+b(t)\right.}$$

$$\lim_{t\to\infty} P\left[\frac{X}{X+Y}\le w,\ \|(X,Y)\|_* > a(t)r+b(t) \,\middle|\, \|(X,Y)\|_* > b(t)\right]$$
$$= J(w)\,\overline{K}(r)$$

J has mass on interior of $[0,1]$, $\overline{K}(r) = \left(1+\frac{\lambda r}{\sigma}\right)_+^{-1/\lambda}$

for some λ, σ.

Special case when $(X, Y) = (X_P, Y_P)$, $a(t) = b(t) = t$,
$\overline{K}(r) = (1+r)^{-1}$.

Then let $R = \|(X,Y)\|_*$, $W = \frac{X}{X+Y}$, $R = (X+Y)\|(W, 1-W)\|_*$

hence $X = \dfrac{RW}{\|(W, 1-W)\|_*}$, $Y = \dfrac{R(1-W)}{\|W, (1-W)\|_*}$.

leads to the model:

$$(A,B) = S \cdot (V_1, V_2), \qquad (V_1,V_2) = \frac{(V, 1-V)}{\|(V,1-V)\|_m} \in \mathcal{S}^m$$

$$\mathcal{S}^m = \{ \underset{\sim}{v} \in \mathbb{R}_+^2 : \|\underset{\sim}{v}\|_m = 1 \}$$

$$S \sim GP(1, \lambda), \qquad V \sim F_V.$$

$\|\cdot\|_m$ and F_V are modeling choices, but λ and pars. of F_V need to be inferred.

Copula:

$$C(u_1, u_2) = F_{A,B} \{ F_A^{\leftarrow}(u_1), F_B^{\leftarrow}(u_2) \}$$

where F_A, F_B are "pseudo marginals"

Inference: assume everything differentiable,

$$c(u_1,u_2) = \frac{f_{A,B}\{ F_A^{\leftarrow}(u_1), F_B^{\leftarrow}(u_2) \}}{f_A\{F_A^{\leftarrow}(u_1)\} f_B\{F_B^{\leftarrow}(u_2)\}} \qquad 0 \le u_1, u_2 \le 1$$

Joint density

$$f_{AB}(x,y) = \frac{\|(x,y)\|_m}{(x+y)^2} \cdot \left\{1+\lambda\|(x,y)\|_m\right\}^{-1/\lambda-1} f_V\left(\frac{x}{x+y}\right)$$

$x, y > 0$

Derivation:
(x,y) generic values of (A,B), (s,v) for (S,V).

$$f_{S,V}(s,v) = (1+\lambda s)_+^{-1/\lambda} \cdot f_V(v).$$

Transform $x = \frac{sv}{q}$, $y = \frac{s(1-v)}{q}$, $q = \|(v,1-v)\|_m$

$$\frac{\partial x}{\partial s} = \frac{v}{q}, \quad \frac{\partial y}{\partial s} = \frac{1-v}{q}, \quad \frac{\partial x}{\partial v} = \frac{s}{q} - \frac{sv}{q^2}\frac{dq}{dv}, \quad \frac{\partial y}{\partial v} = -\frac{s}{q} - \frac{s(1-v)}{q^2}\frac{dq}{dv}$$

$$\frac{\partial y}{\partial s}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial v} = \frac{1-v}{q}\left(\frac{s}{q} - \frac{sv}{q^2}\frac{dq}{dv}\right) - \frac{v}{q}\left(-\frac{s}{q} - \frac{s(1-v)}{q^2}\frac{dq}{dv}\right)$$

$$= \frac{s}{q^2}$$

so $f_{A,B}(x,y) = f_{S,V}(s(x,y), v(x,y)) \Big/ \left|\frac{\partial(x,y)}{\partial(s,v)}\right|$

$$= \left(1+\lambda\|(x,y)\|_m\right)_+^{-1/\lambda-1} f_V\left(\frac{x}{x+y}\right) \frac{\|(x/(x+y), y/(x+y))\|_m^2}{\|x,y\|_m}$$

reduces to given formula

Idea: (A, B) model applies only to extreme values of (A,B), say $\max\{F_A(x), F_B(y)\} > u$, some u close to 1.

Copula $c(u_1, u_2; \xi)$ or $C(u_1, u_2; \xi)$

ξ: parameters in copula.

$\cancel{L(\xi) = \prod_{i:\max(u_{i,1}, u_{i,2}) > u} c(u}$

$$L(\xi) = \prod_{i:\max(u_{1,i}, u_{2,i}) > u} c(u_{1,i}, u_{2,i}; \underset{\sim}{\xi}) \cdot \prod_{i:\text{rest}} C(u, u; \underset{\sim}{\xi})$$

Need to transform data to uniform marginals, e.g. apply prob. integral transform to each.

Specifics: take $V \sim \text{beta}(\alpha, \alpha)$, $\|\cdot\|_m = \|\cdot\|_\infty$, $\lambda \leq 1$

so $\underset{\sim}{\xi} = (\lambda, \alpha)$.

Cases: $\lambda > 0$: $\eta = 1$, $\chi > 0$ (asy. dep.)

$\lambda \to 0^+$: $\chi_\lambda \to 0$
(fixed $\sqrt{\ }$)

$\lambda = 0$: $\eta \le 1$, asymptotic independence

$\lambda < 0$: $\eta < 1$, asymptotic independence

Spatial extremes:

Consider models where we have data from a large number of sites and are interested in the dependence among sites.

Latent process approach

Refs: ~~Coles &~~ Casson & Coles 1999, Russell et al 2020
many papers in between

Basic idea: $Y_t(\underset{\sim}{s}) \sim GEV(\underbrace{\mu_t(\underset{\sim}{s}), \psi_t(\underset{\sim}{s}), \xi_t(\underset{\sim}{s})}_{\text{spatial Gaussian procs.}})$

Maybe additional terms, e.g. (R20),

$$\mu_t(\underset{\sim}{s}) = \theta_1(\underset{\sim}{s}) + SST_t \cdot \theta_2(\underset{\sim}{s})$$

$$\log \psi_t(\underset{\sim}{s}) = \theta_3(\underset{\sim}{s}) + SST_t \cdot \theta_4(\underset{\sim}{s})$$

$$\xi_t(\underset{\sim}{s}) = \theta_5(\underset{\sim}{s})$$

$$\underset{\sim}{\Theta}(s) = \begin{bmatrix} \theta_1(\underset{\sim}{s}) \\ \vdots \\ \theta_5(\underset{\sim}{s}) \end{bmatrix} \text{ is a 5-dim spatial process}$$