Asymptotic dependence and asymptotic independence

General: $P\{X \le x, Y \le y\} = \exp\left[-\left(\frac{1}{x}+\frac{1}{y}\right)A\left(\frac{x}{x+y}\right)\right]$

Pickands fn.

$$P\{X>x, Y>x\} \sim \frac{2\left(1-A\left(\frac{1}{2}\right)\right)}{x} \text{ unless } A\left(\tfrac{1}{2}\right)=1$$

$$\chi = \lim_{x\to\infty} P\{Y>x \mid X>x\} = 2\left\{1-A\left(\tfrac{1}{2}\right)\right\}$$

Aside: assumes unit Fréchet margins for both X and Y. If not, need to transform, but same conclusion.

$\chi = 0$: asymptotically independent

$\chi > 0$: asymptotically dependent

In this case, $\chi = 0$ means X and Y are fully independent, but this may not be true in general.

Example (Ledford & Tawn 1996).

If $(X, Y) \sim$ Normal, correlation ρ, then

$$P(X>x, Y>x) \sim C\, x^{-\frac{2}{1+\rho}} (\log x)^{-\frac{\rho}{1+\rho}}$$

$\chi = 0$ but dependence is non-trivial

New concept!

$$P(X>x, Y>x) \sim \mathcal{L}(x)\, x^{-1/\eta} \qquad 0<\eta\le 1$$

η: "coefficient of tail dependence"

$\mathcal{L}$ is slowly varying: $\frac{\mathcal{L}(tx)}{\mathcal{L}(t)} \to 1$ as $t \to \infty$ for all $x>0$.

Normal case: $\eta = \frac{1+\rho}{2}$

$\frac{1}{2} < \eta < 1$: positive association

$0 < \eta < \frac{1}{2}$: negative

$\eta = \frac{1}{2}$: "near independent".

To estimate η: define $T = \min(X, Y)$

$$P(T > u+t \mid T > u) \sim \left(1 + \frac{t}{u}\right)^{-1/\eta} \quad (u \to \infty)$$

Fit GPD to estimate η (Ledford-Tawn 1996)

Extension to full joint tail (L&T 1997)

Proposed $P(X > x, Y > y) = \mathcal{L}(x,y)\, x^{-c_1} y^{-c_2}$

where $c_1 + c_2 = \frac{1}{\eta}$, $\mathcal{L}$ is bivariate SV in sense

$$g(x,y) = \lim_{t \to \infty} \frac{\mathcal{L}(tx, ty)}{\mathcal{L}(t,t)} \text{ exists}$$

$g(cx, cy) = g(x,y) \ \forall c > 0$, so $g(x,y) = g_*\left(\frac{x}{x+y}\right)$

some ~~g_*~~ g_*

L&T proposed $P(X > x, Y > y) \sim K g_*\left(\frac{x}{x+y}\right) x^{-c_1} y^{-c_2}$

<u>but</u> no clearly defined model for g_*

Comment on terminology: both η and χ have been called "coefficient of tail dependence". Confusing!

Ramos & Ledford (2009, 2011)

Assume $P(X>x, Y>y) = \dfrac{\mathcal{L}(x,y)}{(xy)^{1/(2\eta)}}$

$$\frac{\mathcal{L}(ux, uy)}{\mathcal{L}(u,u)} \to g(x,y) \text{ as } u \to \infty$$

[$\equiv c_1 = c_2$ in L&T theory]

R&L define (S,T) as weak limit of $\left(\frac{X}{u}, \frac{Y}{u}\right)$ given $X>u$, $Y>u$. Then

$$\bar{F}_{ST}(s,t) = P(S>s, T>t) = \frac{g_*(s/(s+t))}{(st)^{1/2\eta}}$$

Let $R = S+T$, $W = \frac{S}{R}$

$$\mu_{RW}(dr, dw) = r^{-(1+1/\eta)}\, dH_\eta(w)$$

H_η some non-neg measure on $[0,1]$

Leads to eqn. (4.55) p.130

Example: $F_{XY}(x,y) = \frac{\lambda_u^{1/\eta}}{N_p}\Big[(\rho x)^{-\frac{1}{\eta}} + \left(\frac{y}{\rho}\right)^{-\frac{1}{\eta}}$

$\cancel{-(\rho x)^{-1/\alpha}}$

$- \left\{ (\rho x)^{-1/\alpha} + \left(\frac{y}{\rho}\right)^{-1/\alpha} \right\}^{\alpha/\eta} \Big]$

Parameters α, ρ, η, asy. dep if $\eta \geq \alpha$, asy. ind. if $\eta < \alpha$

Extension to $p > 2$: Qin, Smith & Ren (2009), Ramos - Ledford (2011).

Applications: example by RLS discussing papers by Herweijer & Seager (2008), Lau & Kim (2012)

Figures on p. 132, 133; Tables on p. 134

Other approaches to MV extremes:

(a) Heffernan-Tawn approach (2004): Focus on limit distribution of $(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_p)$ given $Y_i \to \infty$.

(b) High-dim MV dist's: Dan Cooley