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Extremes in Dependent Sequences

Stationarity:

$$P\{X_{n+1} \le x_1, \ldots, X_{n+k} \le x_k\} = P(X_1 \le x_1, \ldots, X_k \le x_k)$$

same for every $n \ge 0$.

Weakly stationary says $E(X_n) = \mu$,
$\mathrm{Cov}(X_n, X_{n+h}) = \rho_h$ are independent of n.

Need stronger here.

"Strong mixing": exists $g(k)$, $k = 1, \ldots$?

s.t. $g(k) \to 0$ and

$$|P(AB) - P(A)P(B)| \le g(k)$$

for all $A \in \mathcal{F}_{-\infty}^{n}$, $B \in \mathcal{F}_{n+k+1}^{\infty}$, any n

$\mathcal{F}_m^n$: σ-algebra of sets generated by $(X_m, \ldots, X_n)$

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Special case: $g(h)=0$ when $h>m$ is called *m-dependent*

Useful concept: ~~equivalent~~ "associated independent sequence"

$\hat{X}_1, \hat{X}_2 \cdots$ same marginal distributions but independent.

Some of the theory concerns results of the form

$$P(M_n \le u_n) - P(\hat{M}_n \le u_n) \to 0$$

where $M_n = \max(X_1, \ldots, X_n)$

$\hat{M}_n = \max(\hat{X}_1, \ldots, \hat{X}_n)$

Early refs, eg Watson 1954, Loynes 1965.

Alternatively: Gaussian sequences

Berman 1965: for a Gaussian sequence with $r_n \log n \to 0$, same result holds.

But this is not strong mixing.

Need for a unified theory ...

Leadbetter's conditions: $D(u_n)$ and $D'(u_n)$

$D(u_n)$: If $i_1 < \dots < i_p < j_1 < \dots < j_q$, $j_1 - i_p > \ell$,

then

$$\big| P\{X_{i_1} \le u_n, \dots, X_{i_p} \le u_n, X_{j_1} \le u_n, \dots, X_{j_q} \le u_n\}$$

$$- P\{X_{i_1} \le u_n, \dots, X_{i_p} \le u_n\} P\{X_{j_1} \le u_n, \dots, X_{j_q} \le u_n\} \big|$$

$$\le \alpha_{n,\ell}$$

where $\alpha_{n,\ell_n} \to 0$ for some $\ell_n = o(n)$.

Alternatively: $\alpha_{n,[n\lambda]} \to 0$ for each $\lambda > 0$.

Thm 1 Under $D(u_n)$, extremal types theorem holds,
e.g. if $P\{M_n \le a_n x + b_n\} \to G(x)$ and $D(u_n)$
holds for $u_n = a_n x + b_n$, then G must be an EVD.

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Thm 1 does **not** say the limit of $P(M_n \le u_n)$ is same as $P(\hat{M}_n \le u_n)$, only that it's the same type.

Leadbetter's D' condition:

$D'(u_n)$ holds if

$$\cancel{\limsup_{n\to\infty} \sum_{j=2}^{\lfloor n/r \rfloor} P\{X_1 > u_n, X_j > u_n\} = 0}$$

$$\limsup_{n\to\infty} n \sum_{j=2}^{\lfloor n/r \rfloor} P\{X_1 > u_n, X_j > u_n\} \to 0$$

as $r \to \infty$

then $\lim\limits_{n\to\infty} \{P(M_n \le u_n) - P(\hat{M}_n \le u_n)\} = 0$

Interpretation of D': "no clustering" condition

Theorem If $D(u_n)$ and $D'(u_n)$ hold for $u_n = a_n x + b_n$, and G non-degenerate, then

$$P\{M_n \le a_n x + b_n\} \to G(x) \text{ iff } P(\hat{M}_n \le a_n x + b_n) \to G(x)$$

Gaussian sequences: if $\{X_n\}$ is stationary Gaussian and $r_n \log n \to 0$, then $D(u_n)$ and $D'(u_n)$ both hold.

Extremal Index

Suppose D' fails i.e. we do have clusters.

Fix some $\tau \in (0, \infty)$, define $u_n(\tau)$ by

$$P\{X_1 > u_n(\tau)\} \le \frac{\tau}{n} \le P\{X_1 \ge u_n(\tau)\}$$

Under independence: $P(M_n \le u_n) \to e^{-\tau}$

Dependent case: if $D(u_n)$ holds, only possible limits are of form $e^{-\theta\tau}$ for some $\theta \in [0,1]$.

Leadbetter's main result:

For fixed $\tau = \tau_0$, if $D(u_n(\tau_0))$ holds, there exist constant θ and θ' such that

$$\limsup_{n\to\infty} P\{M_n \le u_n(\tau)\} = e^{-\theta\tau}$$
$$\liminf \quad \ldots \quad = e^{-\theta'\tau}$$

for all $\tau \le \tau_0$.

Hence if $P(M_n \le u_n(\tau))$ converges for ~~all~~ least one τ_0, it does for all $\tau \le \tau_0$, and $\theta' = \theta$.

Then θ is called "extremal index"

Alternative definition (O'Brien):

Suppose

(i) $\liminf F^n(u_n) > 0$

or $\liminf P\{\max(X_2, \ldots, X_{p_n}) \le u_n \mid X_1 > u_n\} > 0$

(ii) satisfy $D(u_n)$ (or something slightly weaker)

(iii) $p_n = o(n)$, $\ell_n = o(p_n)$, $n\alpha_{n,\ell_n} = o(p_n)$

then

$$\theta = \lim_n P\{\max(X_2, \ldots, X_{p_n}) \le u_n \mid X_1 > u_n\}.$$

Ex. 1 $\{Z_n\}$ ind. $P\{Z_n \le z\} = e^{-z^{-\alpha}}$ $z \ge 0$, $\alpha > 0$

Let $\{c_j, j \ge 0\}$ be incr., pos. $c_0 = 1$, $c_j \to \infty$,

$$X_n = \max_{j \ge 0} \frac{Z_{n-j}}{c_j}$$

Then $P\{X_n \le x\} = \exp(-Kx^{-\alpha})$, $K = \sum c_n^{-\alpha} < \infty$