 Started on 2/25/25

Dombry-Ferreira Result:

Define $V = \left(-\frac{1}{\log F}\right)^{\leftarrow}$, suppose there exist $\xi_0 \in \mathbb{R}$

and functions $a(t)$, $A(t)$ s.t.

$$\lim_{t\to\infty} \frac{\frac{V(tx)-V(t)}{a(t)} - \frac{x^{\xi_0}-1}{\xi_0}}{A(t)} = H_{\xi_0,\rho}(x), \quad x>0$$

$\cancel{G_\xi(x) = \frac{x^\xi - 1}{\xi}}$, $H_{\xi,\rho}(x)$ as defined previously

$\xi_0 > -\frac{1}{2}$, $\rho \le 0$, $A \in RV(\rho)$

Assume $k = k_n \to \infty$, $m = m_n \to \infty$ s.t.

$$\lim_{n\to\infty} \sqrt{k_n}\, A(m_n) = \lambda \in \mathbb{R}$$

Define $\underset{\sim}{\theta}_0 = (0, 1, \xi_0)$,

$$Q_{\xi_0}(s) = \frac{(-\log s)^{-\xi_0} - 1}{\xi_0} \qquad s \in (0,1)$$

$\cancel{\underset{\sim}{b} + \theta}$ $\underset{\sim}{b}(\xi_0, \rho) = \int_0^1 \frac{\partial^2 \ell}{\partial x \partial \underset{\sim}{\theta}} (\underset{\sim}{\theta}_0, Q_{\xi_0}(s)) \cdot H_{\xi_0,\rho}\left(-\frac{1}{\log s}\right) ds,$

$$I_{\xi_0} = -\int_0^1 \frac{\partial^2 \ell}{\partial \underset{\sim}{\theta} \partial \underset{\sim}{\theta}^T} (\underset{\sim}{\theta}_0, Q_{\xi_0}(s))\, ds$$

Here $\ell = \ell(\mu, \psi, \xi; x) = \log \psi + \log g_\xi\left(\frac{x-\mu}{\psi}\right)$

Then

(a) there exist $\hat{\underset{\sim}{\theta}}_n = (\hat{\mu}_n, \hat{\psi}_n, \hat{\xi}_n)$ s.t.

$$\lim_{n\to\infty} P\{\hat{\underset{\sim}{\theta}}_n \text{ is a MLE}\} = 1,$$

$$\sqrt{k_n}\left(\frac{\hat{\mu}_n - b_{m_n}}{a_{m_n}}, \frac{\hat{\psi}_n}{a_{m_n}} - 1, \hat{\xi}_n - \xi_0\right) \xrightarrow{d} N\left[\lambda I_{\xi_0}^{-1} \underset{\sim}{b}, I_{\xi_0}^{-1}\right]$$

(b) If $\hat{\underset{\sim}{\theta}}^{(i)}$, $i = 1, 2$ are two sequences satisfying the conditions of part (a) and

$$\lim_{n\to\infty} P\left\{\sqrt{k_n}\left(\frac{\hat{\mu}_n^{(i)} - b_{m_n}}{a_{m_n}}, \frac{\hat{\psi}_n^{(i)}}{a_{m_n}} - 1, \hat{\xi}_n^{(i)} - \xi_0\right) \in H_n\right\} =$$

for some $H_n \in \mathbb{R}^3$ ball of radius $r_n = O(k_n^\delta)$, some $0 < \delta \le \min(\frac{1}{2}, \xi_0 + \frac{1}{2})$, then

$$\lim_{n\to\infty} P\{\hat{\underset{\sim}{\theta}}_n^{(1)} = \hat{\underset{\sim}{\theta}}_n^{(2)}\} = 1.$$

STOR834, p. 51 Skip this part for now.

Side Section 1: Heuristic on biased estimation

Sequence of experiments indexed by n.

$X_1, \ldots, X_{k_n} \sim g_n$

In place of true g_n, we assume $f_n(\cdot\,; \underset{\sim}{\theta}_n)$

s.t. $f_n - g_n \to 0$ in some metric as $n \to \infty$

Estimator defined by

$$\sum_{i=1}^{k_n} \underset{\sim}{T}(X_i ; \underset{\sim}{\theta}) = 0$$

and $E\{\underset{\sim}{T}(X_i ; \underset{\sim}{\theta})\} = 0$ when $X_i \sim f_n(\underset{\sim}{\theta})$

Matrix W has entries $w_{rs}(X) = \dfrac{\partial T_r(X)}{\partial \theta_s}$

Assume when $\underset{\sim}{\theta} = \underset{\sim}{\theta}_0$, $E(W) = I_0$ (Fisher inf)

Under regularity conditions,

$$0 = \sum_{i=1}^{k_n} T(X_i ; \hat{\theta}_n) \approx \sum_{i=1}^{k_n} T(X_i ; \underset{\sim}{\theta}_0) + W(X_i ; \underset{\sim}{\theta}_0)(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_0)$$

hence $\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_0 \approx - \left\{ \sum_{i=1}^{k_n} W(X_i ; \theta_0) \right\}^{-1} \cdot \left\{ \sum_{i=1}^{k_n} T(X_i ; \underset{\sim}{\theta}_0) \right\}$

If we assume

(i) $E\{W(X_i ; \theta_0) = J_0\}$ for each i

(ii) $\operatorname{cov} T(X_i ; \underset{\sim}{\theta}_0) = \underset{\sim}{C}_0$ each i,

and if f_n is true density, then

$$\sqrt{k_n}\,(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_0) \xrightarrow{d} N\left[0, J_0^{-1} C_0 J_0^{-1} \right].$$

Now suppose true density is g_n rather than f_n.
Typically:

$$\operatorname{cov}\left\{ \sum_{i=1}^{k_n} \underset{\sim}{T}(X_i ; \theta_0) \right\} \approx k_n C_0(\underset{\sim}{\theta}_0)$$

$$E\left\{ \sum_{i=1}^{k_n} W(X_i ; \theta_0) \right\} \approx k_n J_0(\underset{\sim}{\theta}_0)$$

$$E\left\{ \sum_{i=1}^{k_n} \underset{\sim}{T}(X_i ; \theta_0) \right\} = \underset{\sim}{b}_n \neq 0$$

then
$$k_n^{-1/2} \sum_{i=1}^{k_n} \underset{\sim}{I}(X_i;\theta_0) \sim N\left[k_n^{-1/2} \underset{\sim}{b}_n, C_0(\underset{\sim}{\theta}_0)\right] + o_p(1)$$

$$\sqrt{k_n}(\underset{\sim}{\hat\theta}_n - \underset{\sim}{\theta}_0) + k_n^{-1/2} J_0^{-1}(\underset{\sim}{b}_n) \xrightarrow{d} N\left[0, J_0^{-1} C_0 J_0^{-1}\right]$$

If $k_n^{-1/2} \underset{\sim}{b}_n \to \underset{\sim}{c}$,

$$\sqrt{k_n}(\underset{\sim}{\hat\theta}_n - \underset{\sim}{\theta}_0) \xrightarrow{d} N\left[-J_0^{-1}\underset{\sim}{c}, J_0^{-1} C_0 J_0^{-1}\right]$$

This leads to a "bias-variance tradeoff".

end of skip

Side Section 2: Asymptotics of Hill-Weissman Estimators

Assume $X_1, \ldots, X_n$ where $1-F(x)$ is $RV(\alpha)$ as $x \to \infty$.

Hill's est^n: assume $X_1 \geq X_2 \geq \ldots \geq X_k > u \geq X_{k+1}, \ldots$.

$$\frac{1}{\hat\alpha} = \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_i}{u} \quad \text{for some } u.$$

Question: what is the optimal u?

STOR834 p.54

Assume $1-F(x) = cx^{-\alpha}\left(1+dx^{-\beta}+o(x^{-\beta})\right)$, $x \to \infty$.

$$\frac{1-F(uy)}{1-F(u)} = y^{-\alpha}\left\{1+du^{-\beta}(y^{-\beta}-1)+o(u^{-\beta})\right\}$$

$$\cancel{f_{X_u}(y) = \alpha y^{-\alpha-1} + du^{-\beta}}$$

Density $\alpha y^{-\alpha-1} + du^{-\beta}\left\{(\alpha+\beta)y^{-\alpha-\beta-1} - \alpha y^{-\alpha-1}\right\} + o(u^{-\beta})$

Note integrals of form

$$\int_1^\infty (\log y)^k y^{-\alpha-1}\,dy = \alpha^{-k-1}\,\Gamma(k+1)$$

$= k!$ when k integer

So if $Y_u = \frac{X}{u}$ conditioned on $X>u$, we have

$$E(\log Y_u)^k = \alpha^{-k}k! + du^{-\beta}k!\left\{(\alpha+\beta)^{-k} - \alpha^{-k}\right\} + o(u^{-\beta})$$

$$E\log Y_u = \frac{1}{\alpha} - du^{-\beta}\cdot\frac{\beta}{\alpha(\alpha+\beta)} + o(u^{-\beta})$$

$\therefore$ bias of $\frac{1}{\hat\alpha} \approx -du^{-\beta}\cdot\frac{\beta}{\alpha(\alpha+\beta)}$

end 2/25/25