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Reformulation in terms of inverse functions:

Let $f: \mathbb{R} \to \mathbb{R}$ be nondecreasing. Define

$f^{\leftarrow}(x) = \inf\{y : f(y) \ge x\}$. Left-continuous

[If $f(x)$ is constant y on (a, b), then $f^{-1}(y) = a$]

Define $U(t) = \left(\frac{1}{1-F}\right)^{-1}(t) = \inf\{y : F(y) \ge 1 - \frac{1}{t}\}$.

Remark: In many examples so far, either a_n or b_n is $U(n)$

Consider relations of form

$$\lim_{t \to \infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^{\xi} - 1}{\xi} \text{ for } x > 0.$$

[$\xi = 0$: replace by $\log x$]

Example 1: Suppose $1 - F(x) \sim c x^{-\alpha}$ as $x \to \infty$ $(\alpha > 0)$

Solve $1 - F(x) = \frac{1}{t}$ implies $x \sim (ct)^{\frac{1}{\alpha}}$

$$U(t) = (ct)^{\frac{1}{\alpha}} + o(t^{\frac{1}{\alpha}})$$

Also define $a(t) = \alpha^{-1}(ct)^{\frac{1}{\alpha}}$

so $$\frac{U(tx)-U(t)}{a(t)} = \frac{(ctx)^{\frac{1}{\alpha}}-(ct)^{\frac{1}{\alpha}}+o(t^{\frac{1}{\alpha}})}{\alpha^{-1}(ct)^{1/\alpha}}$$

$$\to \frac{x^{1/\alpha}-1}{1/\alpha} = \frac{x^{\xi}-1}{\xi} \quad \text{with } \xi = \frac{1}{\alpha}.$$

Example 2: Suppose $1-F(x) \sim c|x|^{\alpha}$ as $x \uparrow 0$

$1-F(x) = \frac{1}{t}$ requires $x \sim -(tc)^{-1/\alpha}$

Define $a(t) = \alpha^{-1}(ct)^{-1/\alpha}$

$$\frac{U(tx)-U(t)}{a(t)} = \frac{-(tcx)^{-1/\alpha}+(tc)^{-1/\alpha}+o(t^{-1/\alpha})}{\alpha^{-1}(ct)^{-1/\alpha}}$$

$$\to \frac{x^{-1/\alpha}-1}{(-\alpha^{-1})} = \frac{x^{\xi}-1}{\xi} \quad \text{with } \xi = -\frac{1}{\alpha}.$$

$F(x) = \Phi(x)$. Define b_n by $F(b_n) = 1 - \frac{1}{n}$

implies $U(n) = b_n \sim \sqrt{2\log n}$.

$U(t)$ by interpolation for non-integer t,

$U(t) \sim \sqrt{2\log t}$.

$$U(tx) - U(t) = \sqrt{2\log(tx)} - \sqrt{2\log t} + o(\sqrt{\log t})$$

$$= \sqrt{2\log t}\left\{\left(1 + \frac{\log x}{\log t}\right)^{1/2} - 1 + o(1)\right\}$$

$$\sim \sqrt{2\log t} \cdot \frac{\log x}{2\log t} = \frac{\log x}{\sqrt{2\log t}}$$

so with $a(t) = \frac{1}{\sqrt{2\log t}}$,

$$\frac{U(tx) - U(t)}{a(t)} \to \log x.$$

This is $\xi = 0$ case.

Theorem 2·3 Let F be a cdf and $U(t) = \left(\frac{1}{1-F}\right)^{-1}(t)$ as before. Then there exist constants a_n and b_n s.t. $F^n(a_n x + b_n) \to (1+\xi x)_+^{-1/\xi}$ if and only if there exists a function $a(t)$ such that

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \begin{cases} \frac{x^\xi - 1}{\xi} & \xi \neq 0 \\ \log x & \xi = 0. \end{cases}$$

Side conclusion:
$$\lim_{t\to\infty} \frac{a(tx)}{a(t)} = x^\xi \quad \forall x>0.$$

Called regularly varying of index ξ.

$\xi = 0$ is the slowly varying case.

Second-order approximation

Assume $\frac{U(tx)-U(t)}{a(t)} \to \frac{x^{\xi}-1}{\xi}$ as $t \to \infty$

Next order term: suppose there exists $A(t) \to 0$ such that

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^{\xi}-1}{\xi}}{A(t)} = H(x) \quad (*)$$

First step: exclude $H(x) = c\,\frac{x^{\xi}-1}{\xi}$

because in that case

$$\frac{U(tx)-U(t)}{a(t)} \approx \frac{x^{\xi}-1}{\xi} + cA(t) \cdot \frac{x^{\xi}-1}{\xi}$$

$$\frac{1}{A(t)}\left\{ \frac{U(tx)-U(t)}{a(t)(1+cA(t))} - \frac{x^{\xi}-1}{\xi} \right\} \to 0$$

Theorem (De Haan–Stadtmüller) If $a(t)>0$, $A(t)>0$ and $(*)$ holds with H not a multiple of $\frac{x^{\xi}-1}{\xi}$, then

$$H(t) = c_1 \int_1^x s^{\xi-1} \int_1^s u^{\rho-1}\,du\,ds + c_2 \int_1^x s^{\xi+\rho-1}\,ds, \text{ some } c_1, c_2, \rho \le 0.$$

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Moreover, if this limit exists, we can redefine $a(t)$ and $A(t)$ so that $c_1 = 1$, $c_2 = 0$, then

$$H(t) = \int_1^x s^{\xi-1} \int_1^s u^{\rho-1}\, du\, ds$$

$$= \begin{cases} \frac{1}{\rho}\left(\frac{x^{\xi+\rho}-1}{\xi+\rho} - \frac{x^{\xi}-1}{\xi}\right) & (\rho<0,\ \xi\neq 0) \\ \frac{1}{\xi}\left(x^{\xi}\log x - \frac{x^{\xi}-1}{\xi}\right) & (\rho=0,\ \xi\neq 0) \\ \frac{1}{\rho}\left(\frac{x^{\rho}-1}{\rho} - \log x\right) & (\rho<0,\ \xi=0) \\ \frac{1}{2}(\log x)^2 & (\rho=\xi=0) \end{cases}$$

End 2/13/25

Example 1: (2.11) $1-F(t) = ct^{-\alpha} + dt^{-\alpha-\beta} + o(t^{-\alpha-\beta})$

Solve $1-F(y) = \frac{1}{t} = cy^{-\alpha} + dy^{-\alpha-\beta}\ldots$

$$y = (ct)^{\frac{1}{\alpha}}(1+\varepsilon)$$

$$y^{-\alpha} = \frac{1}{ct}(1-\alpha\varepsilon+\ldots) = y^{-\alpha} + \frac{d}{c}y^{-\alpha-\beta} - \frac{\alpha\varepsilon+.}{ct}$$

so $\frac{\alpha\varepsilon}{ct} \approx \frac{d}{c}y^{-\alpha-\beta}$, $\varepsilon \approx \frac{dt}{\alpha}(ct)^{-1-\beta/\alpha}$

$$u(t) = (ct)^{1/\alpha}\left(1 + \frac{d}{\alpha}c^{-1-\beta/\alpha}t^{-\beta/\alpha} + \ldots\right)$$