 Class of 2/06/25

Beta dist^n

$$f(t; a, b) = \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1} \quad 0<a<1$$

Special case of

$$1 - F(t) = c(\omega_F - t)^\alpha + d(\omega_F - t)^{\alpha+\beta} + o\left((\omega_F - t)^{\alpha+\beta}\right)$$

as $t \uparrow \omega_F$.

In this case $b_n = \omega_F$, $a_n = (nc)^{-1/\alpha}$

$$F^n(a_n x + b_n) \to \exp\left(-|x|^\alpha\right)$$

Expansion leads to error of $o(n^{-\beta/\alpha})$

so long as $\beta < \alpha$.

STOR 834 p.27

Normal distⁿ

Start with: $1-\Phi(x) = \frac{\phi(x)}{x}\left(1 - \frac{1}{x^2} + \frac{3}{x^4} - \frac{15}{x^6} + \cdots\right)$

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\quad \Phi(x) = \int_{-\infty}^{t} \phi(t)\,dt$

$$\frac{1-\Phi(t+x\psi(t))}{1-\Phi(t)} \sim \frac{\phi(t+x\psi(t))}{\phi(t)} \quad \text{if } \frac{\psi(t)}{t} \to 0$$

$$= \exp\left\{-\frac{(t+x\psi(t))^2}{2} + \frac{t^2}{2}\right\}$$

$$= \exp\left\{-tx\psi(t) - \frac{x^2\psi^2(t)}{2}\right\}$$

If $\psi(t) = \frac{1}{t}$, this $\to e^{-x}$ as $t \to \infty$

Gnedenko's condition for Gumbel limit

Define a_n, b_n by $1-\Phi(b_n) = \frac{1}{n}$, $a_n = \frac{1}{b_n}$

Explicit formula for b_n:

$$\frac{1}{n} = \frac{\phi(b_n)}{b_n}\left(1 - \frac{1}{b_n^2} + \frac{3}{b_n^4} - \frac{15}{b_n^6} + \cdots\right)$$

Recap: with $\psi(t) = \frac{1}{t}$, the previous argument shows that

$$\lim_{t \to \infty} \frac{1-\Phi(t+x\psi(t))}{1-\Phi(t)} = e^{-x}$$

If we define b_n so that

$$n(1-\Phi(b_n)) \to 1 \qquad (*)$$

Then with $a_n = \psi(b_n) = \frac{1}{b_n}$,

$$n\{1-\Phi(b_n+a_n x)\} \sim \frac{1-\Phi(b_n+a_n x)}{1-\Phi(b_n)} \to e^{-x}$$

and therefore $n \log \Phi(b_n+a_n x) \to -e^{-x}$

$$\Phi^n(a_n x+b_n) \to e^{-e^{-x}} \quad \text{as before.}$$

The required condition on b_n is $(*)$

Explicit formula?

Use result $1-\Phi(b_n) = \frac{\phi(b_n)}{b_n}\left(1-\frac{1}{b_n^2}+\frac{3}{b_n^4}-\frac{15}{b_n^6}+\dots\right)$

(*) will be satisfied if

$$\frac{n\phi(b_n)}{b_n} \to 1 \qquad (**)$$

First guess: try $n e^{-b_n^2/2} = 1$, leads to

$$b_n = \sqrt{2\log n}$$

But this does not satisfy (**) because we still have b_n in denominator

Next try: $b_n = \sqrt{2\log n} + c_n$

Side remark It is true that $\frac{M_n}{\sqrt{2\log n}} \xrightarrow{P} 1$

where M_n is the max of n iid $N[0,1]$ RVs.

In fact, almost sure convergence also holds, and there is an extension for a law of the iterated logarithm.

With new defn. of b_n consider

$$\frac{\phi(b_n)}{b_n} \sim \frac{1}{\sqrt{2\pi}\cdot b_n} \exp\left\{-\frac{1}{2}\left(\sqrt{2\log n}+c_n\right)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{b_n}\cdot\frac{1}{n}\cdot\exp\left\{-c_n\sqrt{2\log n} - \frac{c_n^2}{2}\right\}$$

Side calculation: need $\exp\left\{-c_n\sqrt{2\log n}-\frac{c_n^2}{2}\right\}$

STOR 834 p.30 $= \sqrt{2\pi}\, b_n$ so $c_n\sqrt{2\log n} \sim -\log(\sqrt{2\pi}\, b_n)$

Ignore second term in exp, set $\sim -\frac{1}{2}\log(4\pi\log n)$

$$c_n\sqrt{2\log n} = -\log\left(b_n\sqrt{2\pi}\right) \qquad \text{as stated}$$

$$c_n = -\frac{\log\sqrt{4\pi\log n}}{\sqrt{2\log n}} = -\frac{(\log 4\pi + \log\log n)}{2\sqrt{2\log n}}$$

suggests

$$b_n = \sqrt{2\log n} - \frac{\log 4\pi + \log\log n}{2\sqrt{2\log n}}$$

In fact this choice of b_n does satisfy (**) and is often cited as the "explicit" formula for b_n, but Hall (1979) showed this does not lead to best rate of convergence.

Hall's definition: solve $\frac{\phi(b_n)}{b_n} = \frac{1}{n}$ exactly

$$1-\Phi\left(b_n+\frac{x}{b_n}\right) = \left(b_n+\frac{x}{b_n}\right)^{-1}\phi\left(b_n+\frac{x}{b_n}\right)\left\{1-\left(b_n+\frac{x}{b_n}\right)^{-2}\right.$$
$$\left.+3\left(b_n+\frac{x}{b_n}\right)^{-4}+\dots\right\}$$

Expand each term as far as $O\left(\frac{1}{b_n^4}\right)$

$$\left(b_n + \frac{x}{b_n}\right)^{-1} = \frac{1}{b_n}\left(1 + \frac{x}{b_n^2}\right)^{-1} = \frac{1}{b_n}\left(1 - \frac{x}{b_n^2} + \frac{x^2}{b_n^4} - \cdots\right)$$

$$\phi\left(b_n + \frac{x}{b_n}\right) = \phi(b_n)\ \exp\left\{-\frac{1}{2}\left(b_n + \frac{x}{b_n}\right)^2 + \frac{b_n^2}{2}\right\}$$

$$= \phi(b_n)\ \exp\left\{-x - \frac{x^2}{2b_n^2}\right\}$$

$$= \phi(b_n)\, e^{-x}\left(1 - \frac{x^2}{2b_n^2} + \frac{x^4}{8b_n^4} - \cdots\right)$$

$$1 - \left(b_n + \frac{x}{b_n}\right)^{-2} + 3\left(b_n + \frac{x}{b_n}\right)^{-4}$$

$$= 1 - \frac{1}{b_n^2}\left(1 + \frac{x}{b_n^2}\right)^{-2} + 3b_n^{-4} = 1 - \frac{1}{b_n^2} + \frac{2x}{b_n^4} + \frac{3}{b_n^4} \cdots$$

$$= 1 - \frac{1}{b_n^2} + \frac{2x+3}{b_n^4} + \cdots$$

Multiply the last three expressions

$$1 - \Phi\left(b_n + \frac{x}{b_n}\right) = \frac{\phi(b_n)}{b_n} e^{-x}\left\{1 - \frac{1}{b_n^2}\left(1 + x + \frac{x^2}{2}\right) + O\left(\frac{1}{b_n^4}\right)\right\}$$

$$n\{1-\Phi(a_n x+b_n)\} = e^{-x}\left\{1-\frac{1}{b_n^2}\left(1+x+\frac{x^2}{2}\right)+O\left(\frac{1}{b_n^4}\right)\right\}$$

Hence $\Phi^n(a_n x+b_n) \to e^{-x}$ with error of $O\left(\frac{1}{b_n^2}\right) = O\left(\frac{1}{\log n}\right)$. Shows error rate.

Can we do better?

Choose μ_n, ψ_n, ξ_n so that

$$e^{-x}\left\{1-\frac{1}{b_n^2}\left(1+x+\frac{x^2}{2}\right)\right\} = \left(1+\xi_n\frac{x-\mu_n}{\psi_n}\right)^{-1/\xi_n}$$

with error of $O(b_n^{-4})$ $(\xi_n \to 0)$

Take log of both sides

$$-x-\frac{1}{b_n^2}\left(1+x+\frac{x^2}{2}\right) = -\frac{1}{\xi_n}\log\left(1+\xi_n\cdot\frac{x-\mu_n}{\psi_n}\right)$$

$$= -\frac{x-\mu_n}{\psi_n}+\frac{\xi_n}{2}\left(\frac{x-\mu_n}{\psi_n}\right)^2 \quad (\text{ignore } O(b_n^{-4}))$$

Check $\mu_n = -\frac{1}{b_n^2}$, $\psi_n = 1-\frac{1}{b_n^2}$, $\xi_n = -\frac{1}{b_n^2}$

With these values,

$$n\{1-\Phi(a_n x+b_n)\} = \left(1+\xi_n \frac{x-\mu_n}{\psi_n}\right)^{-1/\xi_n} + O\left(\frac{1}{b_n^4}\right)$$

$$\Phi^n(a_n x+b_n) = \exp\left\{-\left(1+\xi_n \frac{x-\mu_n}{\psi_n}\right)^{-1/\xi_n}\right\} + O\left(\frac{1}{b_n^4}\right)$$

$$= O\left(\frac{1}{\log^2 n}\right)$$

Conclusion: M_n is approximated by $GEV(\mu_n, \psi_n, \xi_n)$ with error $O\left(\frac{1}{\log^2 n}\right)$.

"Penultimate Approximation"

Refs: Fisher & Tippett (1928)

Hall (1979) for rigorous demonstration of $O\left(\frac{1}{\log n}\right)$ rate

Cohen (1982) for penultimate approx.

Lognormal Dist^n

Suppose $\log X \sim N[\mu, \sigma^2]$. wlog $\mu = 0$, but don't assume $\sigma = 1$

So $F(x) = \Phi(\delta \log x)$ some $\delta > 0$.

Similar calculations: for Gumbel approx. the rate of convergence is $O\left(\frac{1}{\sqrt{\log n}}\right)$ but with GEV approx can get $O\left((\log n)^{-3/2}\right)$

Cohen (1982b) (ref 30 of current text)

In this case we define b_n by $1 - F(b_n) = \frac{1}{n}$,

$\delta \log b_n \sim \sqrt{2 \log n}$, penultimate approx has

$$\xi_n = \frac{1}{\delta^2 \log b_n}\left(1 - \frac{1}{\log b_n}\right) > 0 \text{ for large } n$$

End 2/06/25

An example of a distribution with finite ω_F in the Gumbel domain of attraction:

Consider $1 - F(t) = e^{1/t}$ for $t < 0$.