Other cases are similar, e.g. for (iii),

$$n\{1-F(b_n)\} \to 1,$$

$$\lim_{n\to\infty} n\{1-F(a_n x+b_n)\} = \lim_{n\to\infty} \frac{1-F(b_n+x\psi(b_n))}{1-F(b_n)}$$

$$= e^{-x}$$

Similar results for threshold exceedances, e.g.

assume condition (i), define $\sigma_u = \frac{u}{\alpha}$

$$P\{X-u \geq y\sigma_u \mid X>u\} = \frac{1-F(u+\frac{yu}{\alpha})}{1-F(u)}$$

$$\to \left(1+\frac{y}{\alpha}\right)^{-\alpha} \quad \text{as } u\to\infty.$$

Similar results for cases (ii) and (iii).

Examples

Ex1 t distribution and extensions.

$f(t;\nu) = \sqrt{}$

$$f(t;\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma}$$

$$f(t;\nu) = \frac{1}{\sqrt{\pi\nu}} \cdot \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad t \in \mathbb{R}$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{\nu^{\nu/2}}{\sqrt{\pi}} \left\{ t^{-(\nu+1)} - \frac{\nu(\nu+1)}{2} t^{-(\nu+3)} + O\left(t^{-(\nu+5)}\right) \right\}$$

Integrate term by term:

$$1-F(t;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\nu^{\nu/2-1}}{\sqrt{\pi}} t^{-\nu} \left\{ 1 - \frac{\nu^2(\nu+1)}{2(\nu+2)} t^{-2} + O(t^{-4}) \right\}$$

Suggests general form as $t \to \infty$:

$$1-F(t) = ct^{-\alpha} + dt^{-\alpha-\beta} + o(t^{-\alpha-\beta})$$

where $c>0$, $\alpha>0$, $\beta>0$, $d \in \mathbb{R}$

Set $a_n = (nc)^{\frac{1}{\alpha}}$, $1-F(a_n) = c\cdot(nc)^{-1} = \frac{1}{n}$

$$F^n(a_n x) = e^{-x^{-\alpha}} \text{ for } x>0,$$

proves limit of type (i).

Taylor expansion:

$$n\{1-F(a_n x)\} = x^{-\alpha} + n^{-\beta/\alpha} c^{-1+\beta/\alpha} d x^{-\alpha-\beta} + o(n^{-\beta/\alpha})$$

If $\beta < \alpha$,

$$F^n(a_n x) = \exp(-x^{-\alpha}) \cdot \left\{ 1 - n^{-\beta/\alpha} c^{-1+\beta/\alpha} d x^{-\alpha-\beta} + o(n^{-\beta/\alpha}) \right\}$$

First example of a rate of convergence result.

Questions:

a) Is this expansion uniform in x?

b) Can we ever do better?

Answer to (b) is no, except in case $\beta = 1$.
In that case,

$$n\{1-F(a_n x+b_n)\} = c(a_n x)^{-\alpha}\left(1 - \frac{\alpha b}{a_n x}\right) + d a_n^{-\alpha-1} x^{-\alpha-1} + o(a_n^{-\alpha-1})$$

Set $c\alpha b = d$, central terms cancel

Error is then $o(a_n^{-\alpha-1}) = o(n^{-1/\alpha})$ in $F^n(a_n x+b)$

Argument why $\beta < \alpha$: otherwise, an additional error term of $O(1/n)$. However, this case is not important.

End 2/04/25