STOR 834, p. 20 (1/30/2025)

Gnedenko–de Haan theory:

$$P\left\{\frac{M_n-b_n}{a_n}\le x\right\}=F^n(a_nx+b_n)\to H(x)$$

References: any book on EVT covers these results, but see especially de Haan and Ferreira (2006)

[full text available through canvas "Resources" page]

Basic results (previously given, just a summary)

If there are two such limits, say $F^n(a_nx+b_n)\to H_1(x)$, $F^n(\alpha_nx+\beta_n)\to H_2(x)$

then $\exists\, A>0,\ B$ s.t. $\frac{\alpha_n}{a_n}\to A$, $\frac{\beta_n-b_n}{a_n}\to B$, $H_2(x)=H_1(Ax+B)$

This is Khinchine's Lemma.

Implications for extremes:

$F^{kn}(a_{kn}x+b_{kn})\to H(x)$ but also $F^{kn}(a_nx+b_n)\to H^k(x)$

So $H^k(x)=H(A_kx+B_k)$, H is "max-stable"

Thm 2.1 If H is max-stable, it must be one of the "three types" listed in Chapter 1.

Proof Not given here: see de Haan-Ferreira or many previous publications

Theorem 2.2: domain of attraction

The problem: given a max-stable H, find necessary and sufficient conditions on F so that $F^n(a_n x + b_n) \to H(x)$ for some a_n, b_n.

(i) If $H(x) = \exp(-x^{-\alpha})$, n&s condition is $w_F = \sup\{x : F(x)<1\} = +\infty$ and

$$\lim_{t\to\infty} \frac{1-F(xt)}{1-F(t)} = x^{-\alpha} \text{ for any } x>0.$$

wlog: $b_n = 0$, $a_n = \inf\{x : 1-F(x) \le \frac{1}{n}\}$

(ii) If $H(x) = \exp(-|x|^{\alpha})$ for $x<0$, n&s condition is $w_F < \infty$ and

$$\lim_{t\to 0} \frac{1-F(w_F - xt)}{1-F(w_F - t)} = x^{\alpha}, \text{ any } \alpha>0$$

($t \downarrow 0$)

May define $b_n = w_F$, $a_n = w_F - \inf\{x : 1-F(x) \le \frac{1}{n}\}$

(iii) If $H(x) = \exp(-e^{-x})$, n&s condition is that there exists a function $\psi(t)$, defined for $t < \omega_F$,

s.t. $$\lim_{t \uparrow \omega_F} \frac{1-F(t+x\psi(t))}{1-F(t)} = e^{-x}, \text{ any } x>0.$$

Here $b_n = \inf\{x : 1-F(x) \le \frac{1}{n}\}$, $a_n = \psi(b_n)$.

De Haan's modification: if ψ exists, it may be defined by
$$\psi(t) = \frac{\int_t^{\omega_F}(1-F(s))ds}{1-F(t)}.$$

end 1/30/25

Demonstration that these conditions are sufficient:

(i) With a_n as thus defined, $n\{1-F(a_n)\} \to 1$
Also $a_n \to \infty$

$$\lim_{n\to\infty} n\{1-F(a_n x)\} = \lim_{n\to\infty} \frac{1-F(a_n x)}{1-F(a_n)} = x^{-\alpha}.$$

But we also have $\frac{-\log F(a_n x)}{1-F(a_n x)} \to 1 \quad n \to \infty$

so $\lim_{n\to\infty} n \log F(a_n x) = -x^{-\alpha}$ hence ....