STOR 834 p.17

GPD model: $G(y;\sigma,\xi) = 1-\left(1+\frac{\xi y}{\sigma}\right)^{-1/\xi}$

valid when $1+\xi\frac{y}{\sigma}>0$.

Assume data $Y_1,\ldots,Y_n$, do all the same things

$$\mathcal{I}_1 = \begin{bmatrix} \frac{1}{\sigma^2(1+2\xi)} & \frac{1}{\sigma(1+2\xi)(1+\xi)} \\ \cdot & \frac{1}{(1+2\xi)(1+\xi)} \end{bmatrix}$$

valid when $1+2\xi>0$ ($\mathcal{I}_1 \to \infty$ as $\xi \downarrow -\frac{1}{2}$)

$$\mathcal{I}_1^{-1} = (1+\xi)\begin{pmatrix} 2\sigma^2 & -\sigma \\ -\sigma & 1+\xi \end{pmatrix}$$

Poisson-GPD ~~meod~~ model: add $N \sim \mathrm{Pois}(\lambda)$

1/28/25 MLE for λ is $\hat{\lambda}=N$. Add: Review GEV analysis including Bayesian method

r-largest Model Consider separate examples for
(a) single sample, (b) multiple samples

Single sample with $\xi=0$

$$\ell(\mu,\psi) = r\log\psi + \sum_1^r \frac{Y_i-\mu}{\psi} + \exp\left(-\frac{Y_r-\mu}{\psi}\right)$$
(refer back to p.10)

$$\frac{\partial \ell}{\partial\mu} = -\frac{r}{\psi} + \frac{1}{\psi}\exp\left(-\frac{Y_r-\mu}{\psi}\right)$$

$$\frac{\partial \ell}{\partial\psi} = \frac{r}{\psi} - \sum \frac{Y_i-\mu}{\psi^2} + \frac{Y_r-\mu}{\psi^2}\exp\left(-\frac{Y_r-\mu}{\psi}\right)$$

1st eqn gives $\exp\left(-\frac{Y_r-\mu}{\psi}\right)=r$

2nd eqn:

$$r = \sum \frac{Y_i-\mu}{\psi} - r\left(\frac{Y_r-\mu}{\psi}\right) = \frac{1}{\psi}\sum(Y_i-Y_r)$$

$$\hat{\psi} = \frac{1}{r}\sum(Y_i-Y_r), \quad \hat{\mu} = Y_r + \hat{\psi}\log r \qquad \text{Weissman (1978)}$$

Alternative: let $H(x) = \exp\left[-\left(\frac{x}{\sigma}\right)^{-\alpha}\right]$ $\quad x>0\ \alpha>0\ \sigma>0$

$$h(y_1,\dots,y_r) = \sigma^{-r}\left\{\prod_{i=1}^{r}\alpha\left(\frac{y_i}{\sigma}\right)^{-\alpha-1}\right\}\cdot\exp\left(-\left(\frac{y_r}{\sigma}\right)^{-\alpha}\right)$$

$$l(\sigma,\alpha) = r\log\sigma - r\log\alpha + (\alpha+1)\sum\frac{Y_i}{\sigma} + \left(\frac{Y_r}{\sigma}\right)^{-\alpha}$$

$$\hat{\alpha} = \left\{\frac{1}{r}\sum^{r}(\log Y_i - \log Y_r)\right\}^{-1}$$

$$\hat{\sigma} = r^{1/\hat{\alpha}}\,Y_r$$

Third formulation (not directly EVT, but related ...)

Assume $1-F(x) = cx^{-\alpha}$, $\quad x\ge u$

F left unspecified for $x<u$.

Censored data: assume $Y_1 \ge \dots \ge Y_r > u$ with rest of observations $\le u$.

Density $\prod_{i=1}^{r} (\alpha c Y_i^{-\alpha-1}) \cdot (1-cu^{-\alpha})^{n-r}$

$$\ell = -r\log\alpha - r\log c + (\alpha+1)\sum \log Y_i - (n-r)\log(1-cu^{-\alpha})$$

Likelihood equations

$$\frac{\partial \ell}{\partial \alpha} = -\frac{r}{\alpha} + \sum \log Y_i - \frac{(n-r)cu^{-\alpha}\log u}{1-cu^{-\alpha}}$$

$$\frac{\partial \ell}{\partial c} = -\frac{r}{c} + \frac{(n-r)u^{-\alpha}}{1-cu^{-\alpha}}$$

Solve for MLE:

$$\frac{r}{c} = \frac{(n-r)u^{-\alpha}}{1-cu^{-\alpha}}$$

Subst.:

$$\frac{r}{\alpha} = \sum \log Y_i - \log u \cdot \frac{r}{c}$$

$$\frac{1}{\alpha} = \frac{1}{r}\sum \log \frac{Y_i}{u} \quad \text{or} \quad \hat{\alpha} = \left(\frac{1}{r}\sum \log \frac{Y_i}{u}\right)^{-1}.$$

Hill's estimator, after Hill (1975)

Finish: "Multiple Samples", Venice example, R code

End 1/28/25