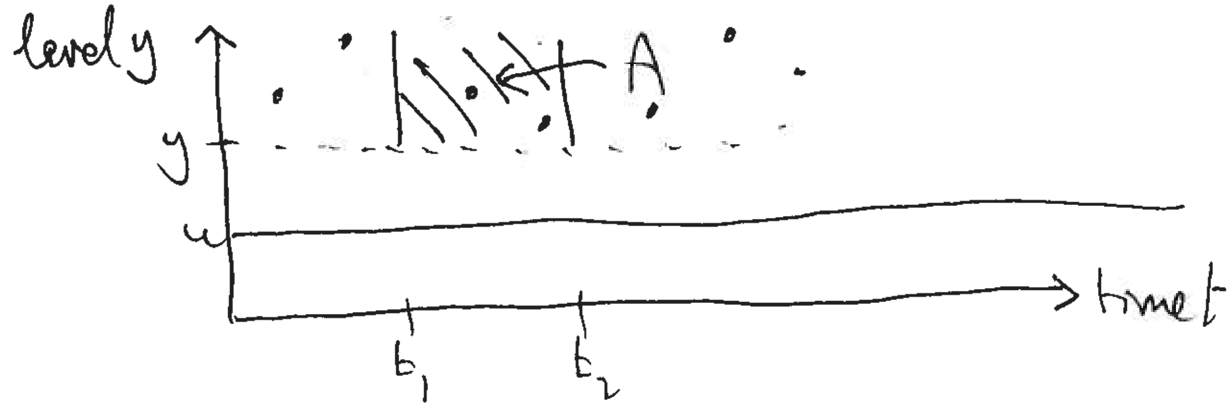
Conclusion: the joint density of $x_1, \ldots, x_{N(A)}$ is proportional to $\prod_{i=1}^{N(A)} \{\lambda(x_i)\} \cdot e^{-\Lambda(A)}$.

Side remark: This result also includes $N(A)=0$ because in that case $P(N(A)=0) = e^{-\Lambda(A)}$.

Back to EVT.

The <u>point process model</u> assumes all exceedances above a fixed threshold u form a NHPP with intensity Λ defined as follows:



For a set A of form $(t_1, t_2) \times (y, \infty)$,

$$\Lambda(A) = (t_2 - t_1) \left(1 + \xi \frac{y-\mu}{\psi}\right)_+^{-1/\xi}$$

We claim each of the GEV, GPD and r-largest representations are special cases of this result.

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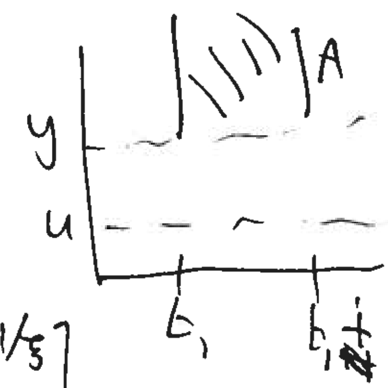
Derivation of GEV from the PPM

Consider an interval of length 1 in some time unit, typically 1 year for environmental data.



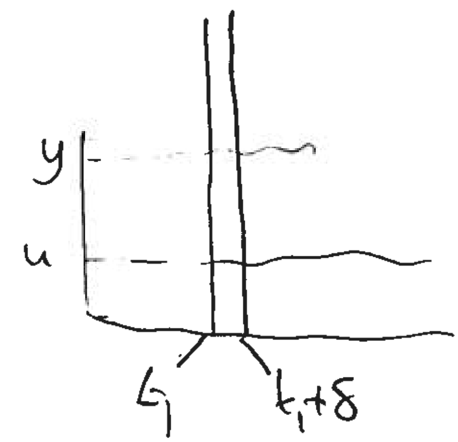
M = max over time (t_1, t_1+1)

$M \le y$ same as A empty

$$P(M \le y) = P(N(A) = 0) = \exp\left[-\left(1+\xi\frac{y-\mu}{\psi}\right)_+^{-1/\xi}\right]$$

is GEV.

Derivation of GPD from the PPM



Consider time interval $(t_1, t_1+\delta)$

Given one obs. $>u$, what is prob. that its $>y$.

Let $A_1 = (t_1, t_1+\delta)\times(u,\infty)$, $A_2 = (t_1, t_1+\delta)\times(y,\infty)$

Find $P(A_2 = 1 \mid A_1 = 1)$

$$= \frac{P(A_2=1)}{P(A_1=1)} = \cancel{e^{-\Lambda(A}} \; \frac{\Lambda(A_2)e^{-\Lambda(A_2)}}{\Lambda(A_1)e^{-\Lambda(A_1)}}$$

$$\approx \frac{\Lambda(A_2)}{\Lambda(A_1)} = \frac{\left(1+\xi\frac{y-\mu}{\psi}\right)^{-1/\xi}}{\left(1+\xi\frac{u-\mu}{\psi}\right)^{-1/\xi}}$$

$$= \left\{ \frac{\psi + \xi(y-\mu)}{\psi + \xi(u-\mu)} \right\}^{-1/\xi}$$

$$= \left\{ \frac{\psi + \xi(u-\mu) + \xi(y-u)}{\psi + \xi(u-\mu)} \right\}^{-1/\xi}$$

$$= \left\{ 1 + \frac{\xi(y-u)}{\psi + \xi(u-\mu)} \right\}^{-1/\xi}$$

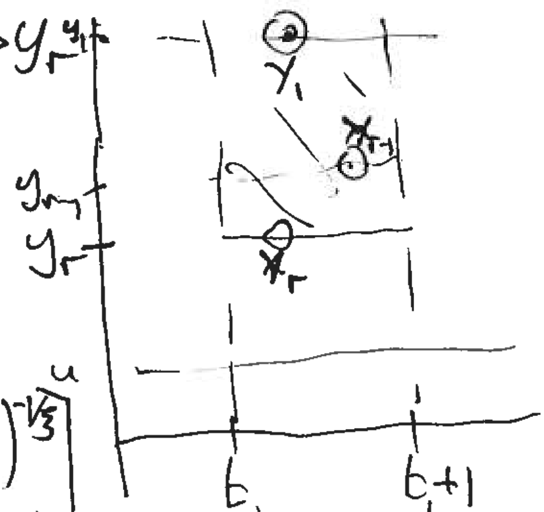
$$= \left(1 + \frac{\xi(y-u)}{\sigma_u} \right)^{-1/\xi} \quad \text{defining } \sigma_u = \psi + \xi(u-\mu)$$

This is of GPD form

Derivation of r largest model from the PPM

Suppose time interval (t_i, t_i+1), consider prob. that r largest events are at $y_1 > y_2 > \ldots > y_r$

Prob that $N(A) = r$ with points $(T_1, Y_1) \ldots (T_r, Y_r)$ at $(t_1, y_1) \ldots (t_r, y_r)$



$$\prod_{i=1}^{r} \left(1 + \xi \frac{y_i - \mu}{\psi} \right)^{-1/\xi} \cdot \exp\left[- \left(1 + \xi \frac{y_r - \mu}{\psi} \right)^{-1/\xi} \right]$$

conditioned on $1 + \xi \frac{y_i - \mu}{\psi} \geq 0$ each $i = 1, \ldots, r$

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Estimation Main method: MLE
Others: Bayes, PWM, L-moments

Consider GEV: Data $Y_1, \dots, Y_n$

$$H(y; \mu, \psi, \xi) = \exp\left[-\left(1+\xi\frac{y-\mu}{\psi}\right)^{-1/\xi}\right]$$

defined when $1+\xi\frac{y-\mu}{\psi} > 0$

Let $h(y; \mu, \psi, \xi) = \frac{\partial H}{\partial y}$

$$= \frac{1}{\psi}\left(1+\xi\frac{y-\mu}{\psi}\right)^{-1/\xi-1} \exp\left[-\left(1+\xi\frac{y-\mu}{\psi}\right)^{-1/\xi}\right]$$

$$-\log h = \log\psi + \left(\frac{1}{\xi}+1\right)\log\left(1+\xi\frac{y-\mu}{\psi}\right) \cdot \left(1+\xi\frac{y-\mu}{\psi}\right)^{-1/\xi}$$

Log likelihood:

$$\ell(\mu, \psi, \xi \mid Y_1, \dots, Y_n) = \sum_{i=1}^{n}\left\{-\log(Y_i; \mu, \psi, \xi)\right\}$$

$$= \sum_{i=1}^{n}\left\{\log\psi + \left(\frac{1}{\xi}+1\right)\log\left(1+\xi\frac{Y_i-\mu}{\psi}\right) + \left(1+\xi\frac{Y_i-\mu}{\psi}\right)^{-1/\xi}\right\}$$

Feasibility condition: $1+\xi\frac{Y_i-\mu}{\psi} > 0$ for all i

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Likelihood equations:

$$\frac{\partial \ell}{\partial \mu} = \frac{\partial \ell}{\partial \psi} = \frac{\partial \ell}{\partial \xi} = 0$$

In practice solved by minimizing ℓ (optim or nlm in R)

Remarks

(i) Local max may not be unique and may not be the global max

specifically if $\xi < -1$ and $1 + \xi \frac{Y_{max} - \mu}{\psi} \to 0$, $\ell \to -\infty$.

However in practice we usually assume $|\xi| < 1$.

(ii) Find "return level"

e.g. $H(y; \mu, \psi, \xi) = \frac{1}{T}$ T given, solve for y

Solution:

$$RV_T = \mu + \psi \frac{\{-\log(1-1/T)\}^{-\xi} - 1}{\xi}$$

Example: Jenkinson's data for Hartford, p. 16 of text

Jenkinson claimed: $\hat{\mu} = 19.68$ $\hat{\psi} = 3.48$ $\hat{\xi} = -0.258$

My calculations: 19.6809 $\hat{\psi} = 3.4788$ $\hat{\xi} = -0.2575$

Also $\widehat{RV}_{1000} = 30.9$ (Jenkinson); I got 30.9105

$\widehat{RV}_{\infty} = 33.2$ 33.193