STOR 834 1/14/2025

Examples from Chapter 1:

(a) Extreme weather events. Plots from Kelowna, Heathrow and Houston. Statistical questions about outliers, trends.

Trying to assess probabilities of such extreme events suggestion: add °F to plots

(b) Insurance example.

393 claims, 7 claim types.

Largest 10: 776.2, 268.0, 142.0, 131.0, ...

shows influence of very large claims

Plots — Fig 1.3

Questions:

1. Distribution of largest claims
2. Any evidence of a trend?
3. Differences among claim types
4. Future projections

(c) Women's track times from 1993

Fig 1.4: 10 best-times per year in 1500 m and 3000 m track events, 1974–1993

Trying to assess prob. of new record conditional on past events

Same question with 2024 Chepngetich example

Overview of Univariate Extremes

Three Types Theorem (Fisher–Tippett, Gnedenko, de Haan)

$X_1, X_2, \ldots$ IID random variables with distribution function $F(x) = P\{X_i \le x\}$, $x \in \mathbb{R}$.

Define ω_F the right-hand endpoint

$$\omega_F = \sup\{x : F(x) < 1\} \le \infty$$

$$M_n = \max(X_1, \ldots, X_n)$$

$$P\{M_n \le x\} = F^n(x)$$

If $F(x) < 1$ then $P\{M_n \le x\} \to 0$ as $n \to \infty$

Also write $M_n \to_p \omega_F$.

Not interesting.

Instead consider normalizing: find constants $a_n > 0$, b_n such that

$$P\left\{\frac{M_n - b_n}{a_n} \le x\right\} = F^n(a_n x + b_n) \to H(x) \quad (*)$$

for some nondegenerate distribution function H.

Question: What H's are possible limits?

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Three Types Theorem: If H exists, it must be one of the three types

$$H(x) = \exp(-e^{-x}) \quad \text{all } x \in \mathbb{R}$$

Gumbel

$$H(x) = \begin{cases} 0 & x < 0 \\ \exp(-x^{-\alpha}) & x > 0 \end{cases} \quad \text{some } \alpha > 0$$

Fréchet

$$H(x) = \begin{cases} \exp(-|x|^{\alpha}) & x < 0 \\ 1 & x > 0 \end{cases} \quad \text{some } \alpha > 0$$

Weibull

Meaning of "type": two distribution functions H and H_1 are of same type if

$$H_1(x) = H(Ax+B), \quad \text{some } A > 0, \ B \in \mathbb{R}.$$

Rationale for considering type: if one $H(x)$ can arise as the limit in (*), so can $H(Ax+B)$ for any $A > 0, B$.

Mini-proof Suppose (*) holds. Given A and B.

Let $a_n' = a_n A$, $b_n' = a_n B + b_n$.

$$F^n(a_n' x + b_n') = F^n(a_n A x + a_n B + b_n)$$
$$= F^n(a_n(Ax+B) + b_n) \to H(Ax+B)$$

Background about Weibull distribution: usually written $F(x) = ~~\text{Next}~~\ 1 - \exp\left\{-\left(\frac{x}{\sigma}\right)^{\alpha}\right\}$ for $x > 0$

or sometimes $F(x) = 1 - \exp\left\{-\left(\frac{x-\beta}{\sigma}\right)^{\alpha}\right\}$, $x > \beta$

where α, σ (and β?) are unknown parameters

Used in strength of materials, running times, etc.

Generalized Extreme Value (GEV) Distribution

$$H(x) = \begin{cases} \exp\left\{-\left(1 + \xi \frac{x-\mu}{\psi}\right)_{+}^{-1/\xi}\right\} & \xi \neq 0 \\ \exp\left[-\exp\left(-\frac{x-\mu}{\psi}\right)\right\} & \xi = 0 \end{cases}$$

where $\left(1 + \xi \frac{x-\mu}{\psi}\right)_{+}$ means $\max\left(1 + \xi \frac{x-\mu}{\psi}, 0\right)$

Motivation of $\xi = 0$ case:

Relies on $\lim_{\xi \to 0} (1 + \xi y)^{-1/\xi} = e^{-y}$.

Cases: $\xi > 0$, $\xi = 0$, $\xi < 0$ correspond to Fréchet, Gumbel, Weibull examples

If $\xi > 0$, equivalent to Fréchet type with $\alpha = +\frac{1}{\xi}$

$$1 - H(x) \propto x^{-1/\xi} \quad \text{as} \quad x \to \infty$$

$$1 + \xi \frac{x-\mu}{\psi} > 0 \quad \text{iff} \quad x > \mu - \frac{\psi}{\xi}$$

Distribution with finite lower bound, no upper bound.

If $\xi < 0$, equivalent to Weibull type with $\alpha = -\frac{1}{\xi}$

$$1 + \xi \frac{x-\mu}{\psi} > 0 \quad \text{iff} \quad x < \mu - \frac{\psi}{\xi}$$

Finite upper bound, no lower bound

So, define $\omega_H = \mu - \frac{\psi}{\xi}$ in this case,

consider $x = \omega_H - y = \mu - \frac{\psi}{\xi} - y$

$$1 + \xi \frac{x-\mu}{\psi} = 1 + \frac{\xi}{\psi}\left(- \frac{\psi}{\xi} - y \right) = - \frac{\xi y}{\psi} = \frac{y}{\alpha \psi}$$

$$1 - H(x) \propto \left(1 + \xi \frac{x-\mu}{\psi} \right)^{-1/\xi} = \left(\frac{y}{\alpha \psi} \right)^{\alpha} \quad \text{as } y \to 0$$

Origins: von Mises (1936), Jenkinson (1955, 1969)

Suggested addition to text: add plots for GEV with $\xi > 0$, $\xi = 0$, $\xi < 0$