This class is often called the *Fréchet class of distributions* (Fréchet (1927)).

(b) The distribution function with $\gamma = 0$,

$$G_0(x) = \exp\left(-e^{-x}\right),$$

for all real x, is called the *double-exponential* or *Gumbel distribution*.

(c) For $\gamma < 0$ use $G_\gamma(-(1+x)/\gamma)$ and get with $\alpha = -1/\gamma > 0$,

$$\Psi_\alpha(x) := \begin{cases} \exp\left(-(-x)^\alpha\right), & x < 0, \\ 1, & x \geq 0. \end{cases}$$

This class is sometimes called the *reverse-Weibull class of distributions*.

Recall that if relation (1.1.1) holds with $G = G_\gamma$ for some $\gamma \in \mathbb{R}$, we say that the distribution function F is in the domain of attraction of G_γ. Notation: $F \in \mathcal{D}(G_\gamma)$.

The result of Theorem 1.1.3 leads to the following reformulation of Theorem 1.1.2.

Theorem 1.1.6 *For $\gamma \in \mathbb{R}$ the following statements are equivalent:*

1. *There exist real constants $a_n > 0$ and b_n real such that*

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G_\gamma(x) = \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \tag{1.1.19}$$

for all x with $1 + \gamma x > 0$.

2. *There is a positive function a such that for $x > 0$,*

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a(t)} = D_\gamma(x) = \frac{x^\gamma - 1}{\gamma}, \tag{1.1.20}$$

where for $\gamma = 0$ the right-hand side is interpreted as $\log x$.

3. *There is a positive function a such that*

$$\lim_{t\to\infty} t\left(1 - F(a(t)x + U(t))\right) = (1+\gamma x)^{-1/\gamma}, \tag{1.1.21}$$

for all x with $1 + \gamma x > 0$.

4. *There exists a positive function f such that*

$$\lim_{t\uparrow x^*} \frac{1 - F(t + xf(t))}{1 - F(t)} = (1+\gamma x)^{-1/\gamma} \tag{1.1.22}$$

for all x for which $1 + \gamma x > 0$, where $x^* = \sup\{x : F(x) < 1\}$.

Moreover, (1.1.19) *holds with $b_n := U(n)$ and $a_n := a(n)$. Also,* (1.1.22) *holds with $f(t) = a\left(1/(1 - F(t))\right)$.*

Proof. The equivalence of (1), (2), and (3) has been established in Theorem 1.1.2. Next we prove that (2) implies (4).

It is easy to see that for $\varepsilon > 0$,