4/4/19
Notes written in class

$$\underset{n\times 1}{y} = \underset{n\times p}{X}\underset{p\times 1}{\beta} + \underset{n\times q}{Z}\underset{q\times 1}{\gamma} + \underset{n\times 1}{\varepsilon}$$

$$\varepsilon \sim N[0, \sigma_\varepsilon^2 I_n]$$

$$Ey = X\beta$$

$$\gamma \sim N[0, \phi\sigma_\varepsilon^2 D]$$

$$\mathrm{cov}(y) = E[(y-Ey)(y-Ey)^T]$$

$$= E[(Z\gamma+\varepsilon)(Z\gamma+\varepsilon)^T]$$

$$E[Z\gamma\gamma^T Z^T + Z\gamma\varepsilon^T + \varepsilon\gamma^T Z^T + \varepsilon\varepsilon^T]$$

$$= Z \cdot D\sigma^2 \cdot Z^T + \sigma^2 I_n$$

$$= \sigma^2 \cdot (I_n + ZDZ^T)$$

$$y \sim N[X\beta, V\sigma^2]$$

$$V = I_n + ZDZ^T$$

Case 1: V known

$$y \sim N[X\beta, V\sigma^2]$$

Density of y is

$$f(y \mid \beta, \sigma^2, V)$$
$$= (2\pi\sigma^2)^{-\frac{n}{2}} |V|^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(y-X\beta)^T V^{-1}(y-X\beta)\right]$$

$|V|$: determinant of V.

$\ell = \log f$

$$\ell = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2}\log|V| - \frac{1}{2\sigma^2}(y-X\beta)^T V^{-1}(y-X\beta)$$

Estimate β:

Minimize $(y-X\beta)^T V^{-1}(y-X\beta)$

("generalized least squares")

Ordinary LS ($V=I$): $\hat{\beta} = (X^TX)^{-1}X^Ty$

Generalized LS: $\hat{\beta} = (X^TV^{-1}X)^{-1}X^TV^{-1}y$

Estimation of σ^2:

$$S = (y - X\hat{\beta})^T V^{-1} (y - X\hat{\beta})$$

write τ $\cancel{\phi} = \sigma^2$

Minimize wrt τ $\quad \frac{n}{2}\log\tau + \frac{1}{2\tau}\cdot S$

$$\frac{n}{2\tau} - \frac{S}{2\tau^2} \Rightarrow \hat{\tau} = \frac{S}{n}$$

$$\hat{\sigma}^2 = \cancel{\frac{1}{n}} \cdot \frac{(y - X\hat{\beta})^T V^{-1} (y - X\hat{\beta})}{n(-p)}$$

The MLE is $\frac{(y - X\hat{\beta})^T V^{-1} (y - X\hat{\beta})}{n}$

However, the unbiased estimator is

$$\hat{\sigma}^2 = \frac{(y - X\hat{\beta})^T V^{-1} (y - X\hat{\beta})}{n-p}$$

So there's a disadvantage of using the MLE, but for now, we still use MLE.

$$V = I + ZDZ^T$$

$$l_1 = -\frac{n}{2}\log\hat{\sigma}^2 - \frac{1}{2}\log|V| - \frac{(y-X\hat{\beta})^T V^{-1}(y-X\hat{\beta})}{2\hat{\sigma}^2}$$

$$= -\frac{n}{2}\log\hat{\sigma}^2 - \frac{1}{2}\log|V| - \frac{n}{2}$$

based on (biased) MLE

MLE:

min ~~max~~imize wrt D $\left(\frac{n}{2}\log\hat{\sigma}^2 + \frac{1}{2}\log|V|\right)$

This should have been minimize

$\Rightarrow$ Maximum likelihood estimator (MLE)

Alternative: REML estimator

$$l_2 = -\frac{n}{2}\log\hat{\sigma}^2 - \frac{1}{2}\log|V|$$

$$\hat{\sigma}^2 = \frac{(y-X\hat{\beta})^T V^{-1}(y-X\hat{\beta})}{n-p}$$

Make $\hat{\sigma}^2$ the unbiased est^n.

Note $n-p$

$$l_2 = \cancel{-\frac{n}{2}\log} - \frac{n-p}{2}\log\hat{\sigma}^2 - \frac{1}{2}\log|V|$$

This should be a minus sign.

$$+\frac{1}{2}\log|X^T V^{-1} X|$$