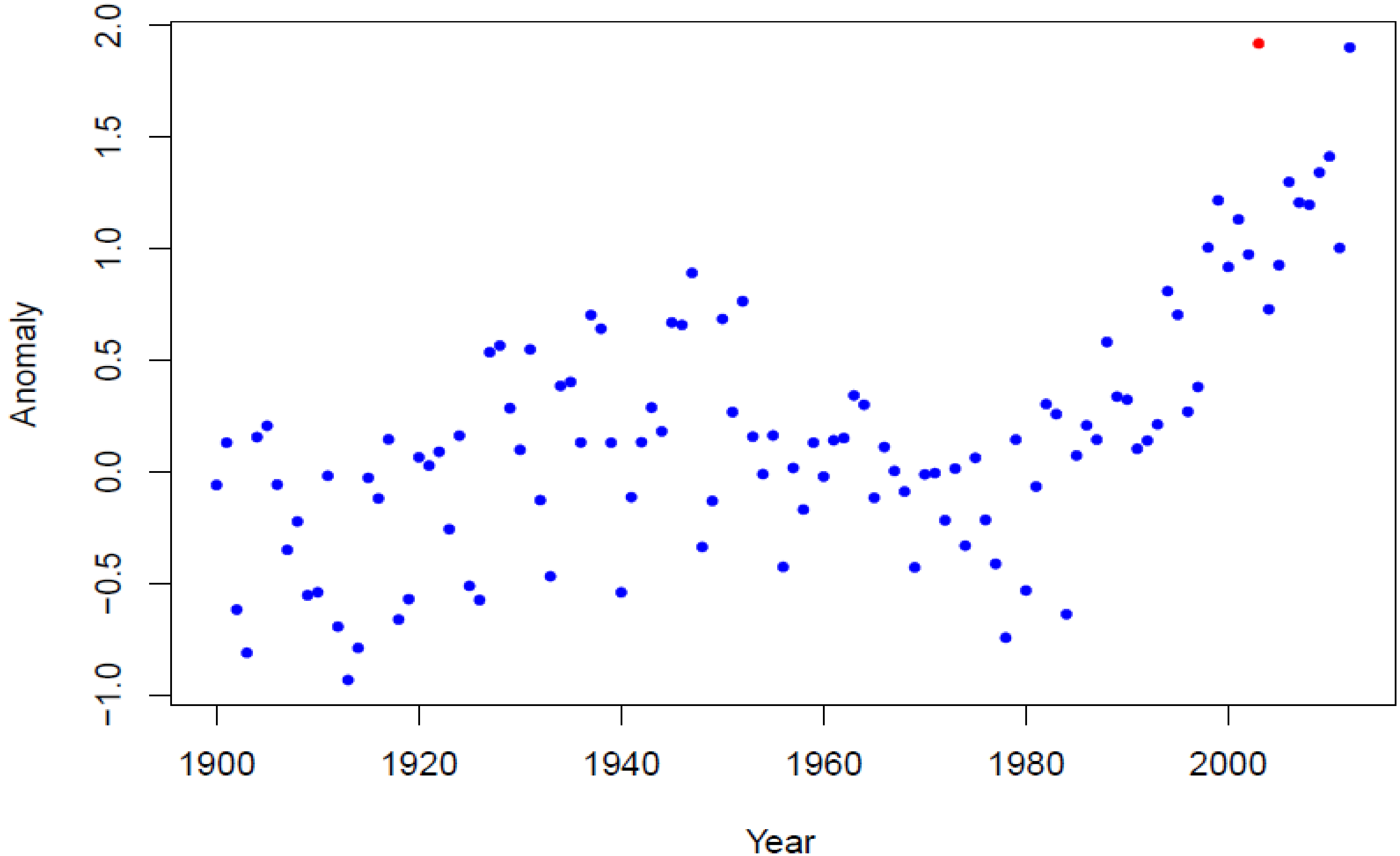
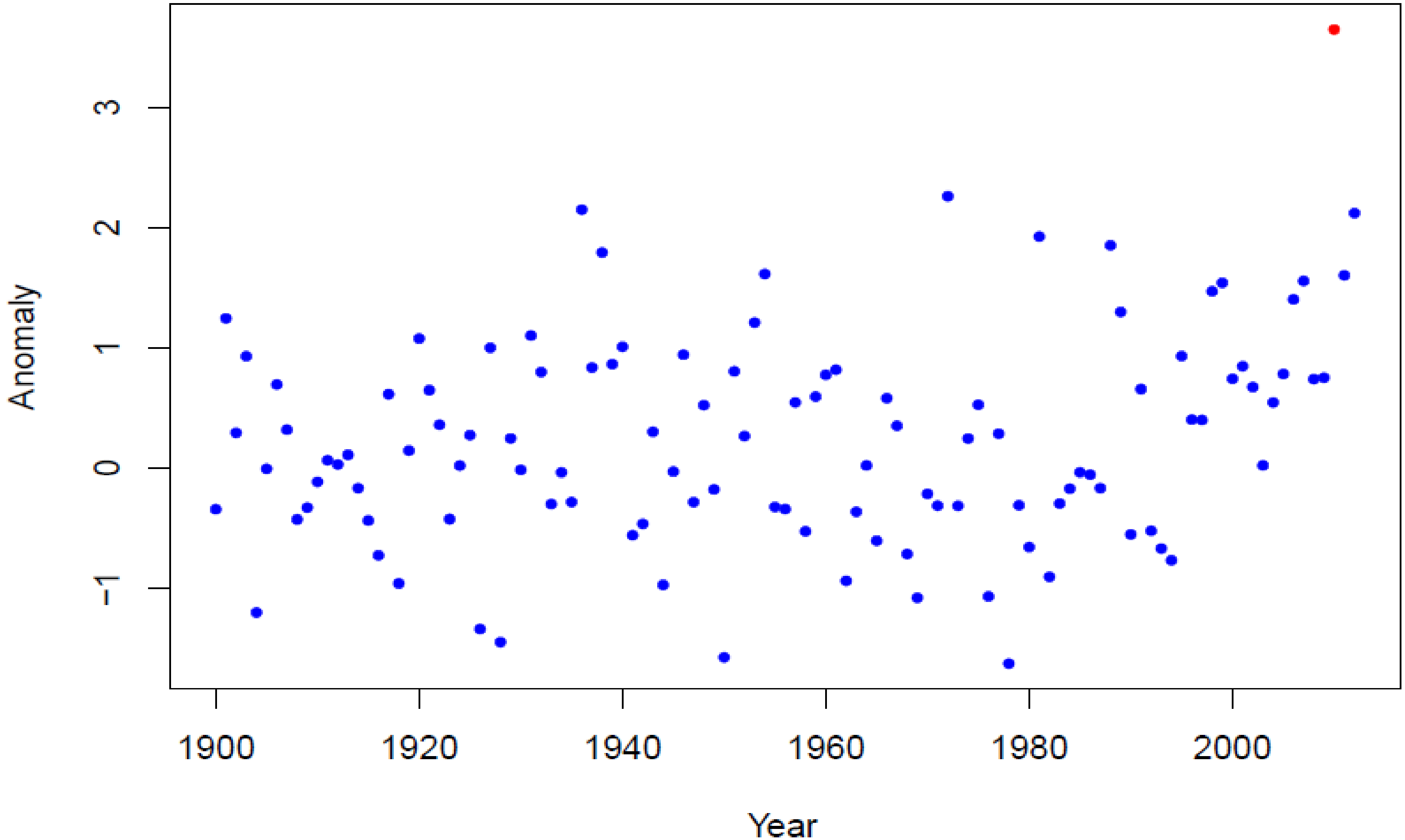
Data

- Observational data from CRU (Climate Research Unit, University of East Anglia, UK) – monthly averages on 5^ox5^o grid boxes, aggregated to JJA average anomalies over
  - Europe: spatial averages over 10^oW-40^oE, 30^oN-50^oN (2003 value was 1.92K but 2012 almost the same)
  - Russia: spatial averages over 30^oE-60^oE, 45^oN-65^oN (2010 value 3.65K)
  - Central USA (including Texas and Oklahoma): spatial averages over 90^oW-105^oW, 25^oN-45^oN (2011 value 2.01K)
- Climate model data from CMIP3
  - 14 climate models
  - Total of 64 control runs, 44 twentieth century runs, 34 future projections under A2 scenario
  - Same spatial regions as observational data, converted to anomalies

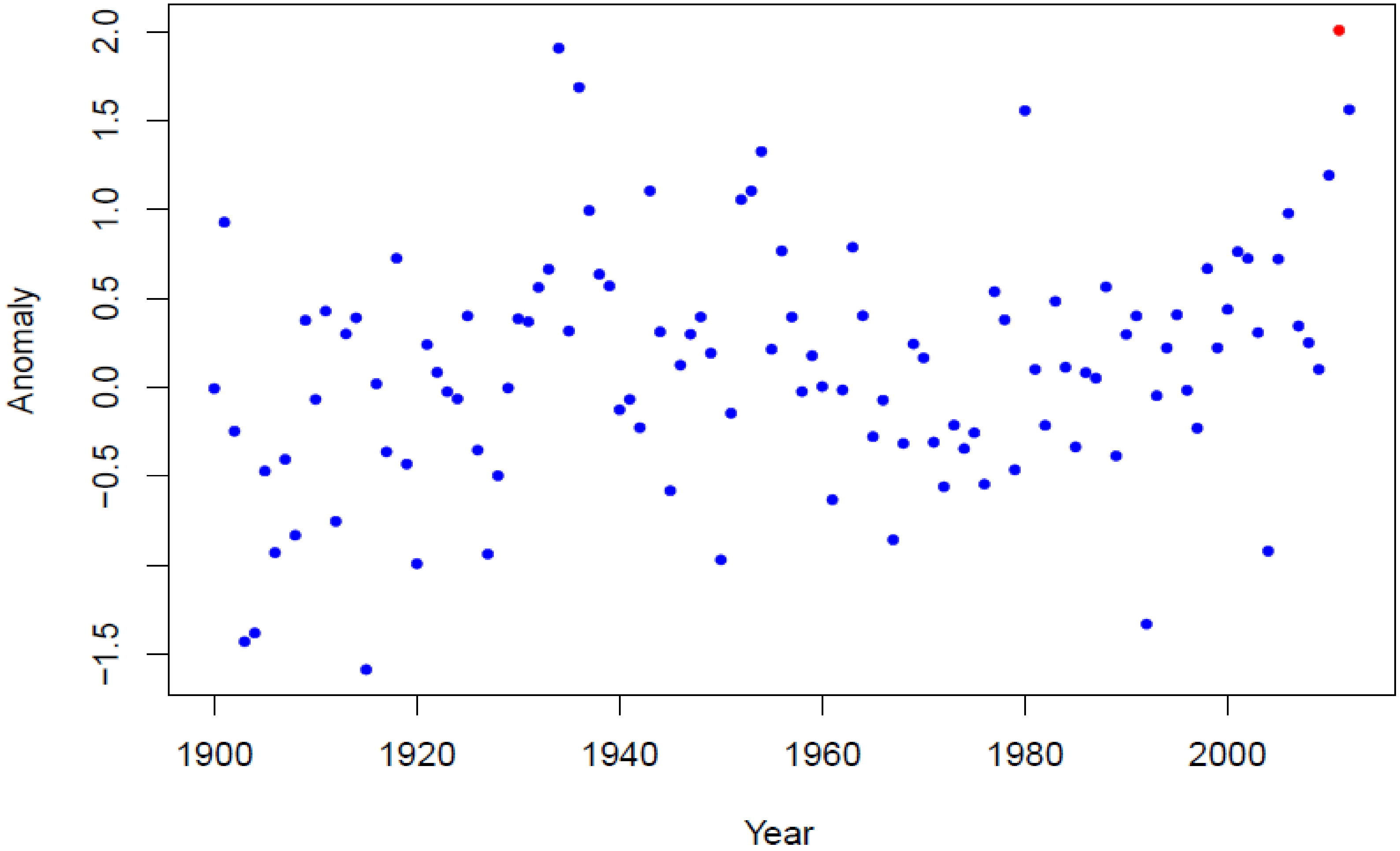
Europe Summer Mean Temperatures



Russia Summer Mean Temperatures



Central USA Summer Mean Temperatures



Introduction To Extreme Value Theory

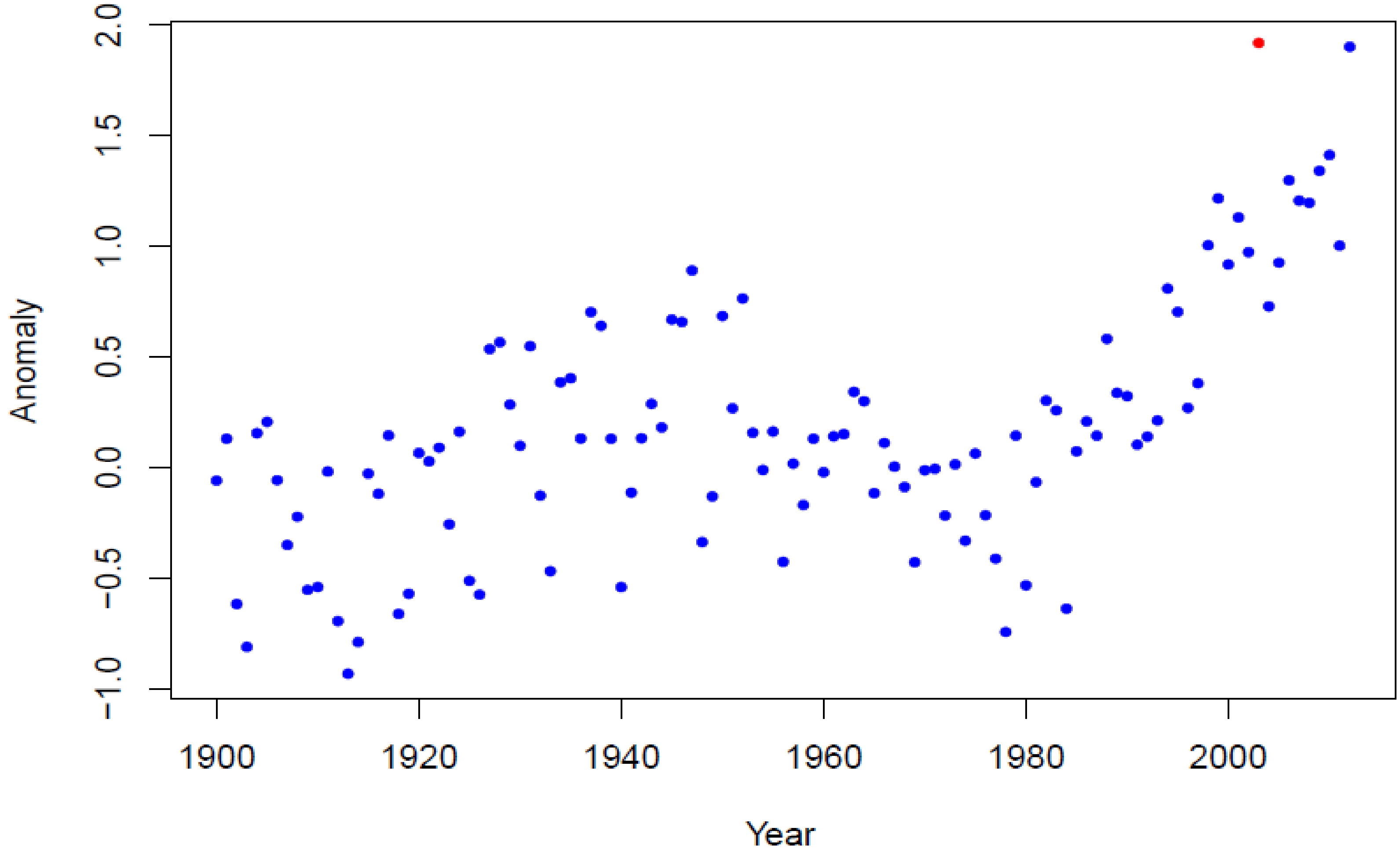
Key tool: *Generalized Extreme Value* Distribution (GEV)

- Three-parameter distribution, derived as the general form of limiting distribution for extreme values (Fisher-Tippett 1928, Gnedenko 1943)
- μ, σ, ξ known as location, scale and shape parameters
- $\xi > 0$ represents long-tailed distribution, $\xi < 0$ short-tailed

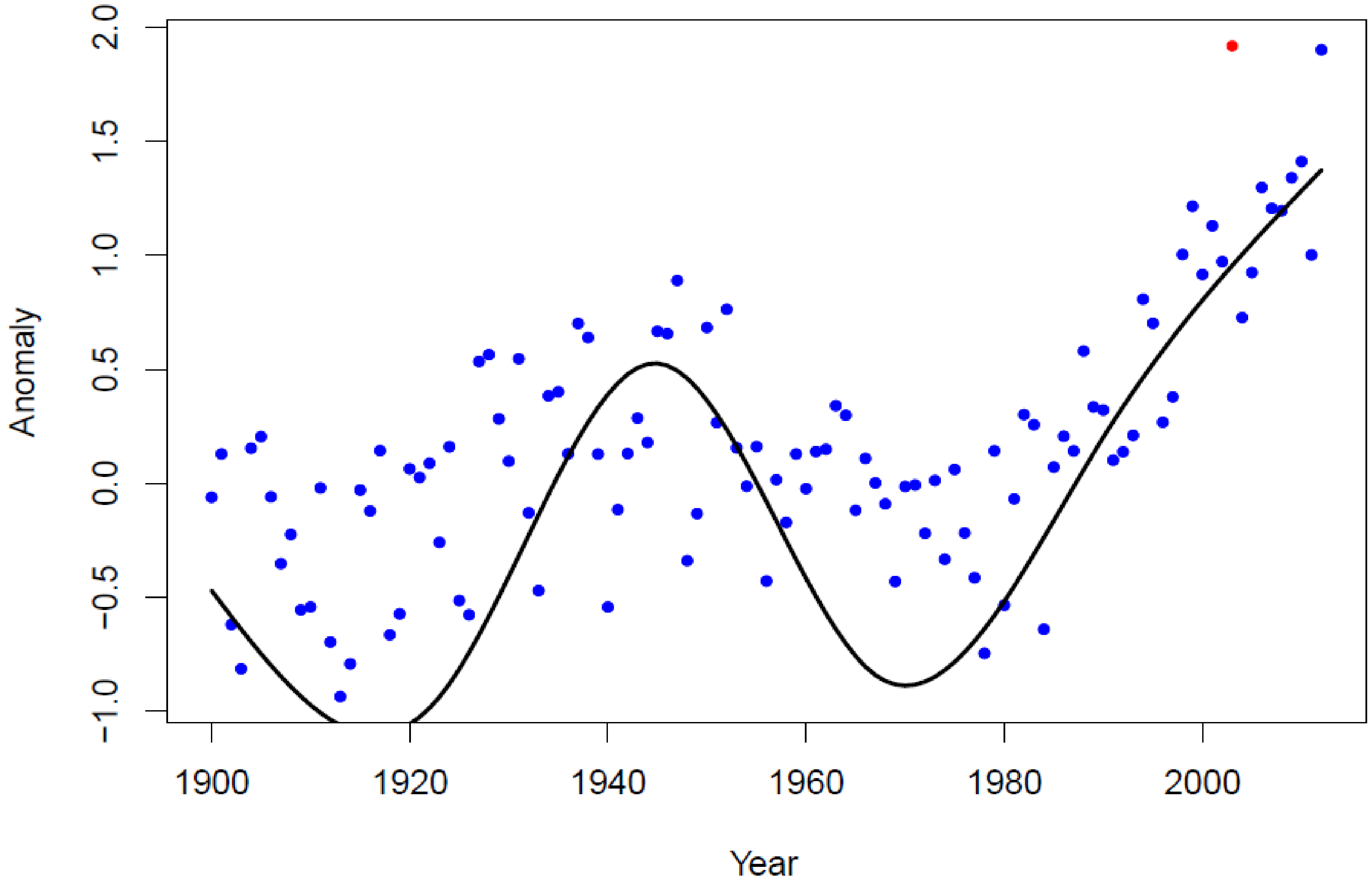
$$\mathrm{Pr}\{Y \le y\} = \exp\left[-\left\{1+\xi\left(\frac{y-\mu}{\sigma}\right)\right\}_{+}^{-1/\xi}\right].$$

- *Peaks over threshold* approach implies that the GEV can be used generally to study the tail of a distribution: assume GEV holds exactly above a threshold u and that values below u are treated as left-censored
- Time trends by allowing μ, σ, ξ to depend on time
- *Example:* Allow $\mu_t = \beta_0 + \sum_{k=1}^{K} \beta_k x_{kt}$ where $\{x_{kt},\ k = 1, ..., K,\ t = 1, ..., T\}$ are spline basis functions for the approximation of a smooth trend from time 1 to T with K degrees of freedom
- Critical questions:
  - Determination of threshold and K
  - Point and interval estimates for the probability of exceeding a high value, such as 1.92K in the case of the Europe time series

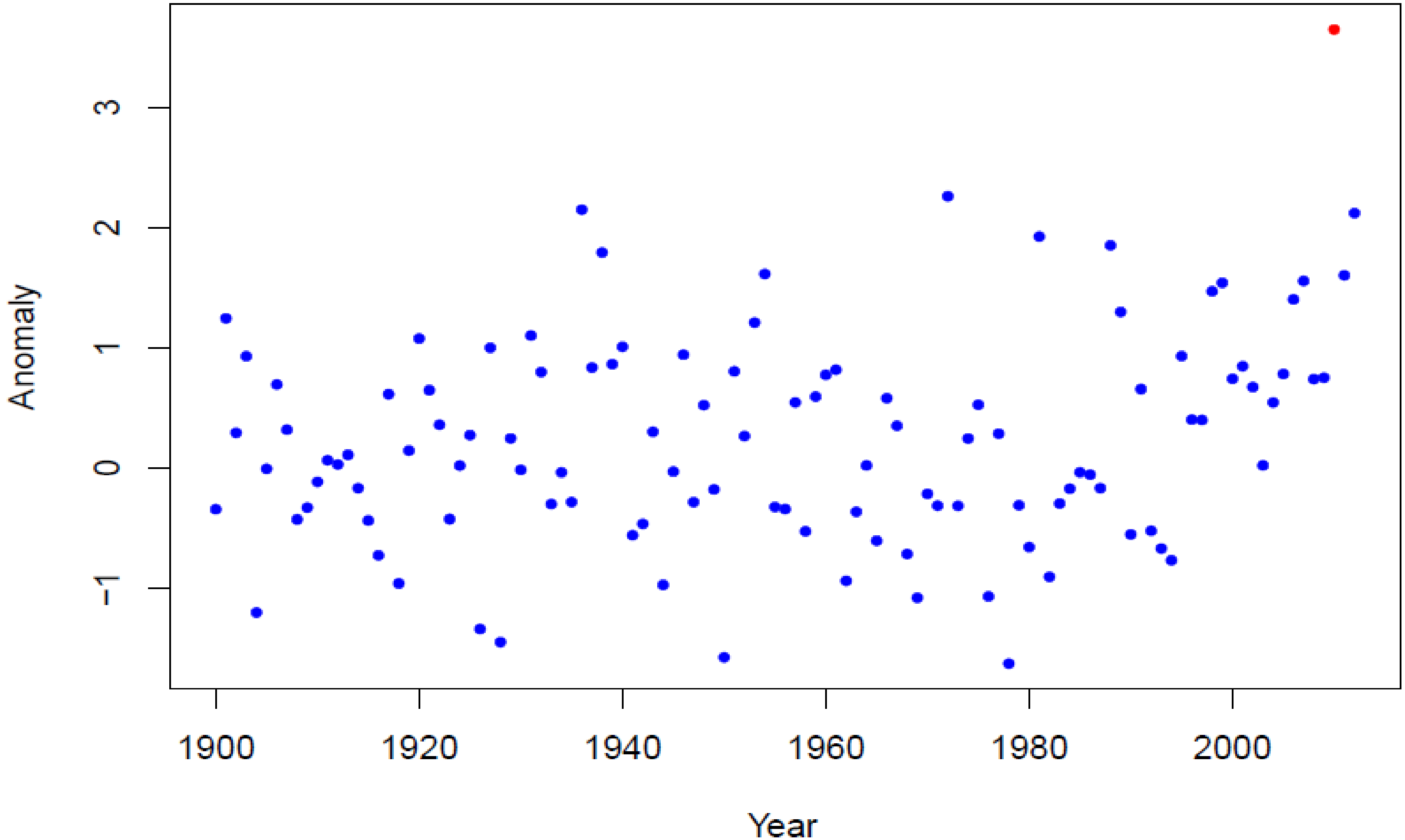
Europe Summer Mean Temperatures



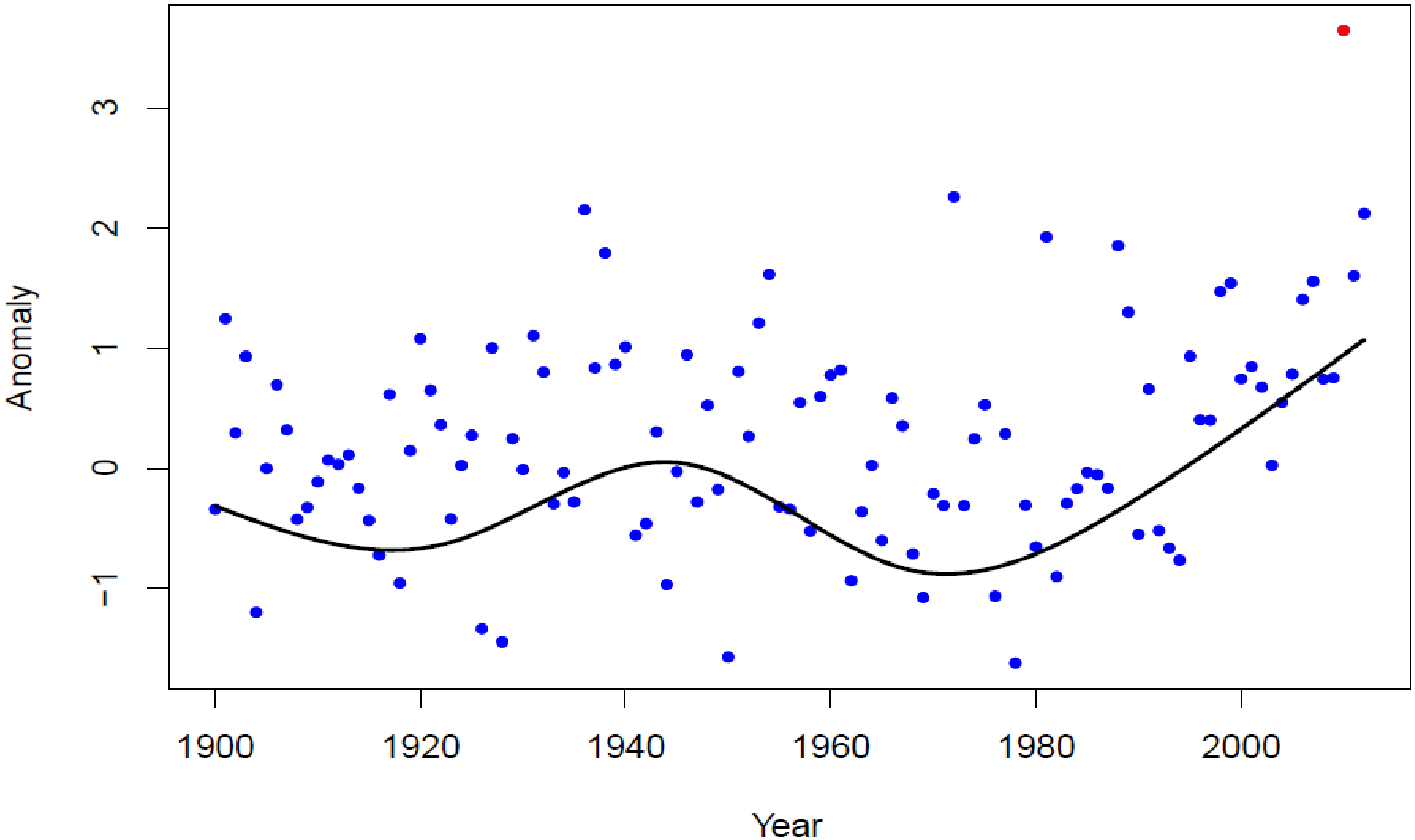
Europe Summer Mean Temperatures With Trend



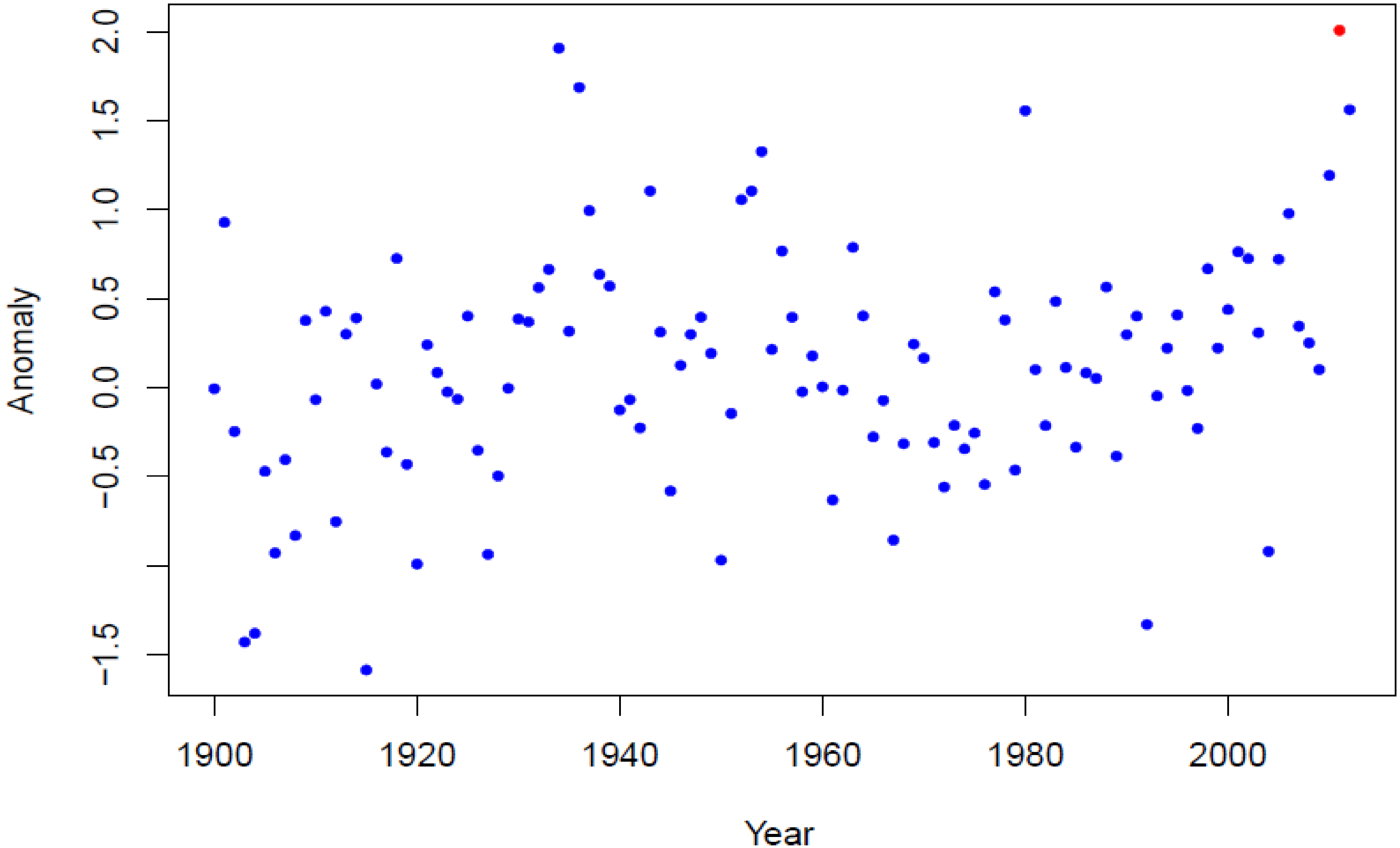
Russia Summer Mean Temperatures



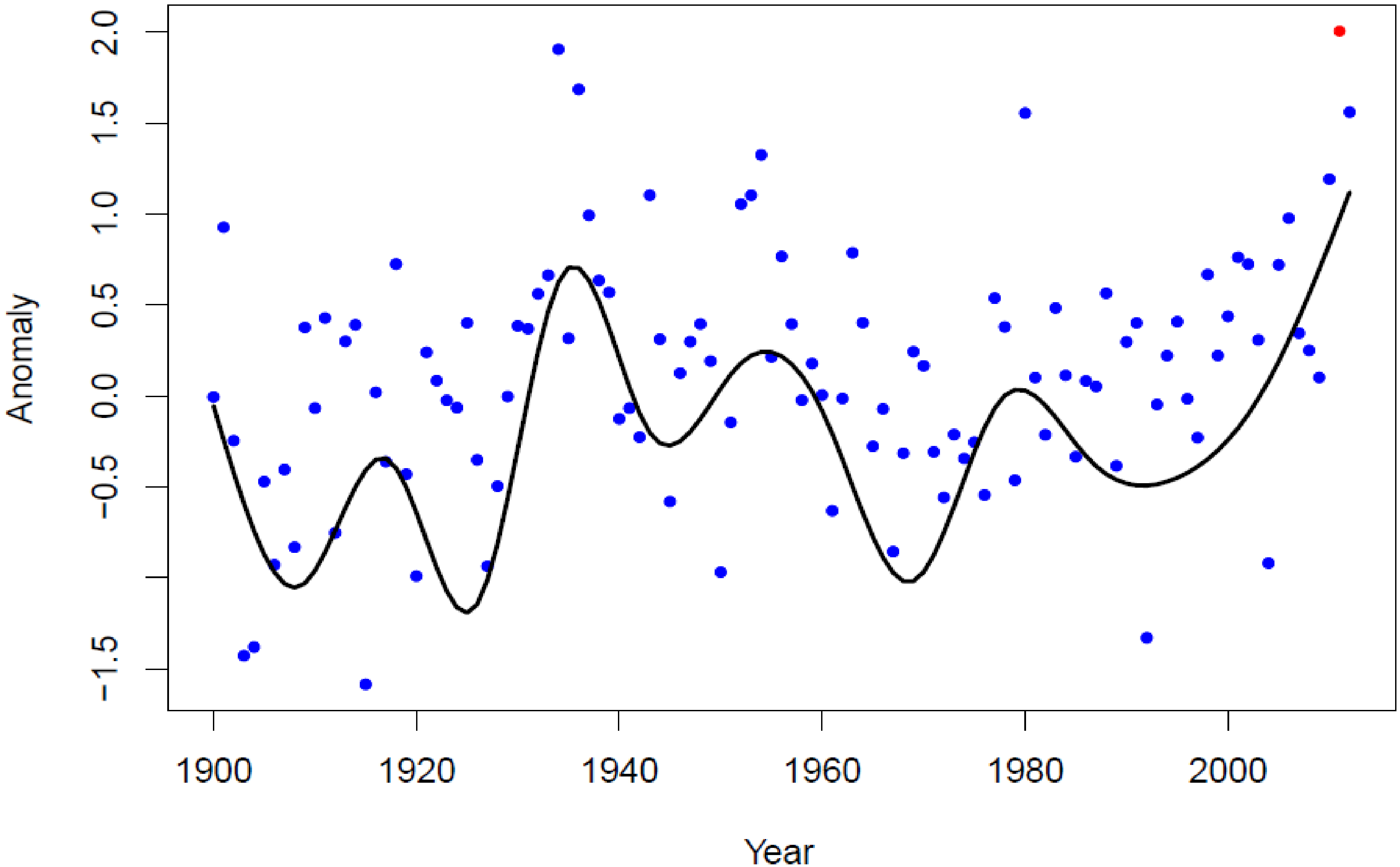
Russia Summer Mean Temperatures With Trend



Central USA Summer Mean Temperatures

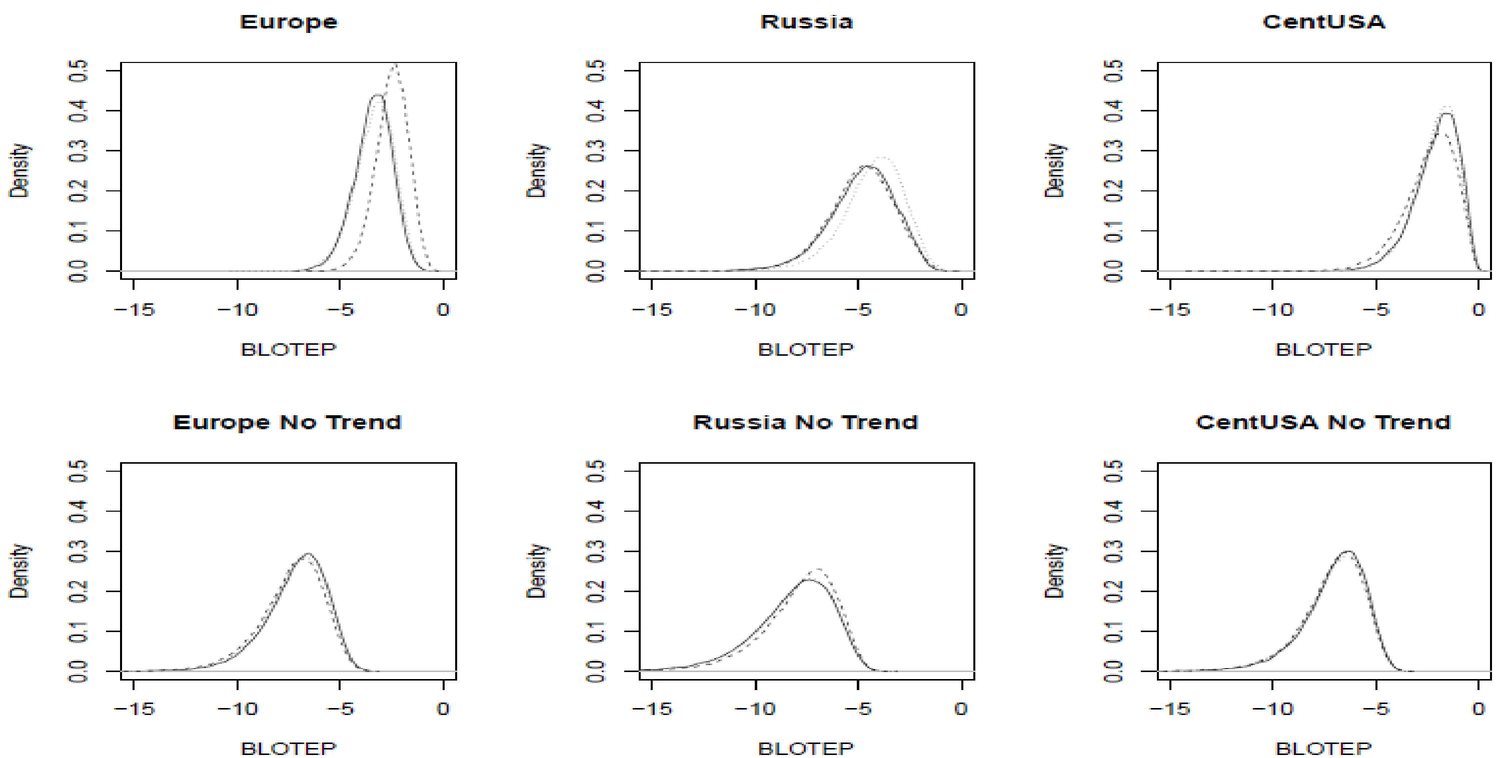


Central USA Summer Mean Temperatures With Trend



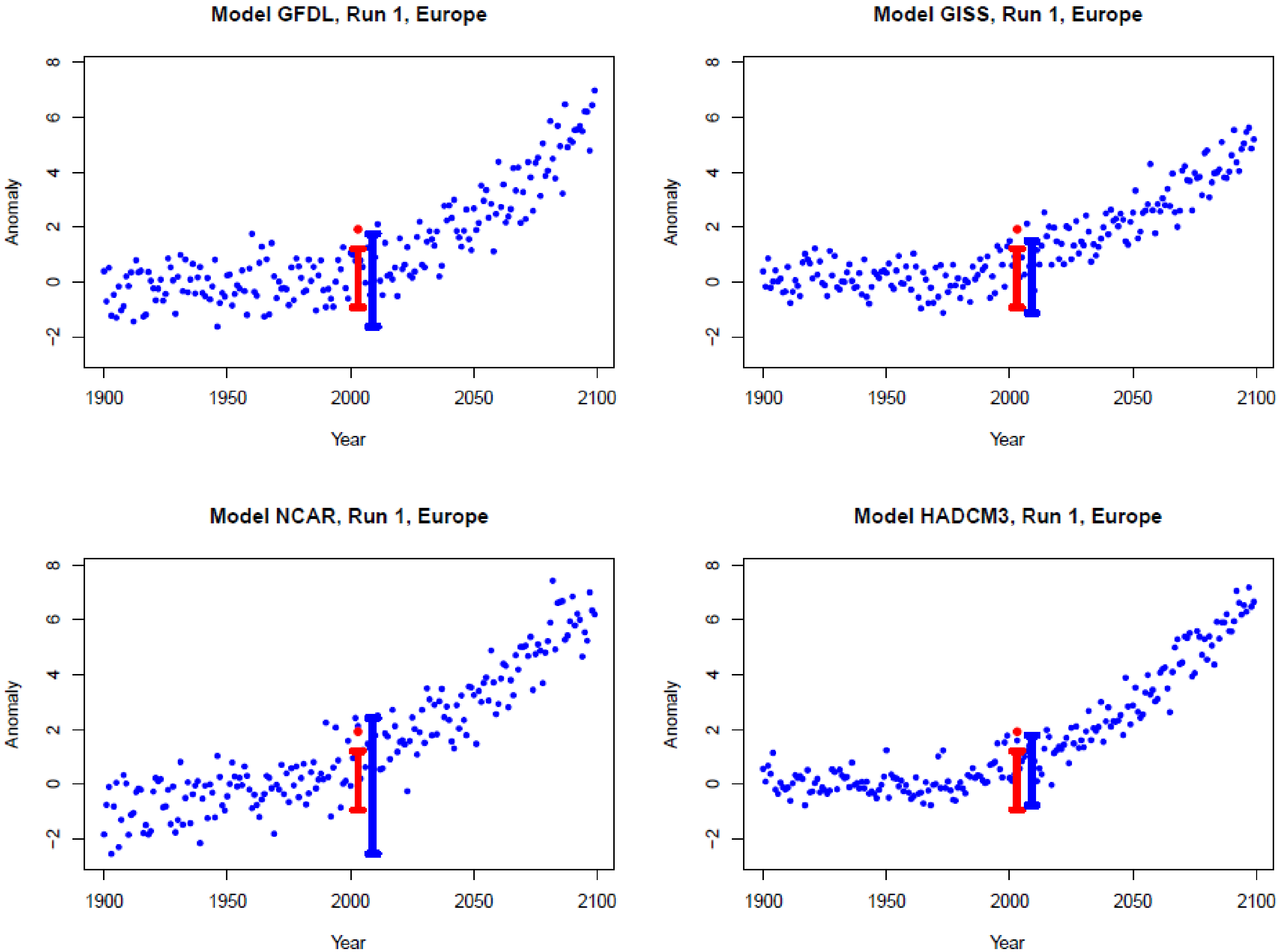
Bayesian Calculations

- Focus on posterior distribution of binary log of threshold exceedance probability (BLOTEP)
- Use models both with and without trends
- Use 80th (solid curve), 75th (dashed) and 85th (dot-dashed) percentiles for thresholds

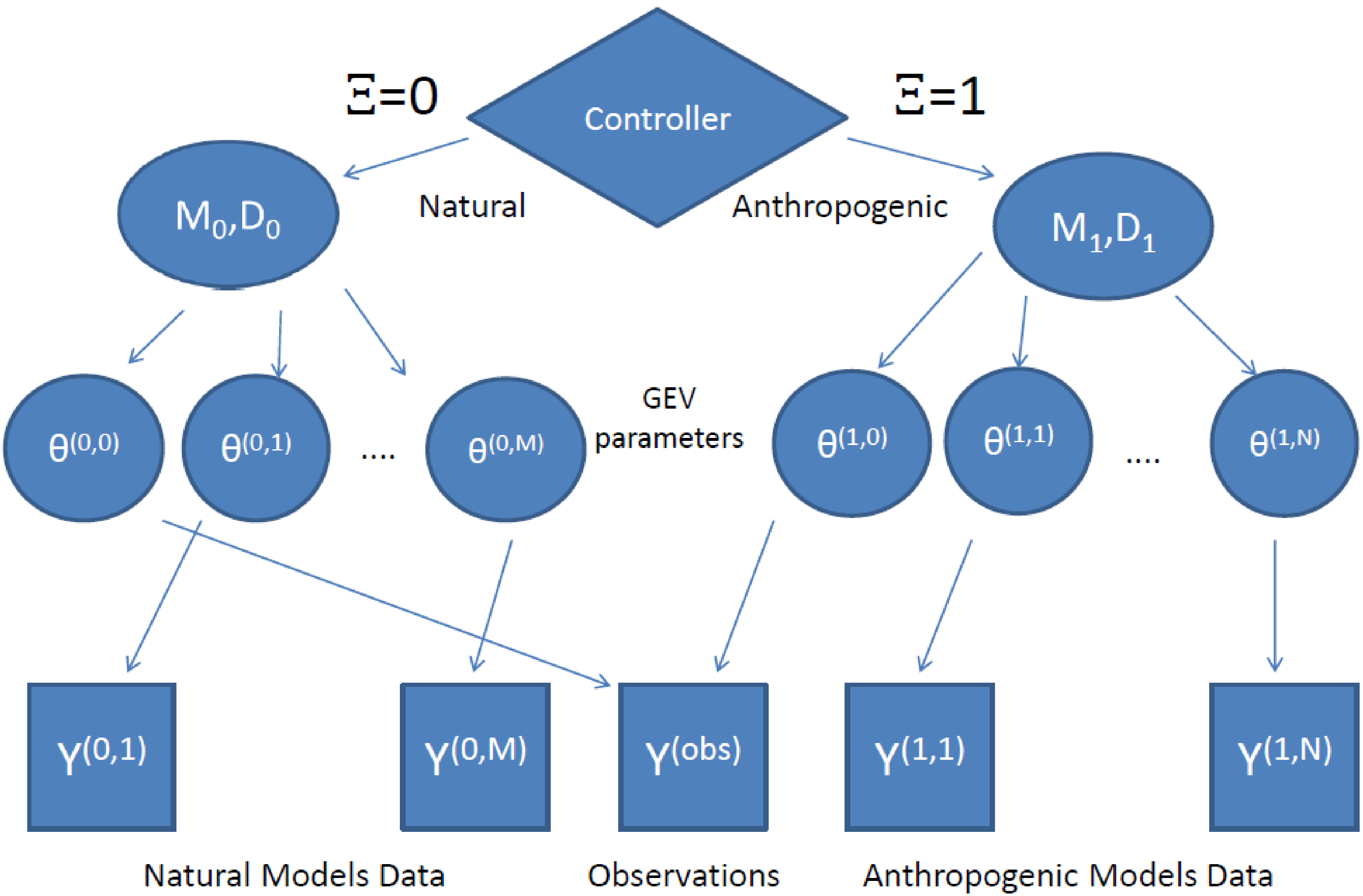


What's Next?

- Obvious strategy at this point is to rerun the GEV calculation on the model data
- But this runs into the *scale mismatch problem*: data plots shows that the models and observations are on different scales, so we should expect the extreme value parameters to be different as well
- Requires a more subtle approach – *hierarchical modeling*



Proposed Hierarchical Model



Bayesian Statistics Details

Model Specification

- $(M_1, D_1) \sim WN_q(A, m, M^*, F)$, Wishart-Normal prior with density $\propto |D_1|^{(m-q)/2} \exp\left[-\frac{1}{2}\mathrm{tr}\left\{D_1\left(A + F(M_1 - M^*)(M_1 - M^*)^T\right)\right\}\right]$.
- Given M_1, D_1, $\theta^{(1,0)}, ..., \theta^{(1,N)}$ are IID $\sim N_q(M_1, D_1^{-1})$.
- Given $\theta^{(1,j)}$, $\mathbf{Y}^{(1,j)}$ generated by GEV with parameters $\theta^{(1,j)}$ ($\mathbf{Y}^{(\mathrm{obs})}$ for $j = 0$, if $\Xi = 1$)
- Similar structure for M_0, D_0 etc.
- We can expand this model by defining $\theta^{(1,0)} \sim N_q(M_1, (\psi D_1)^{-1})$ where ψ represents departure from exchangeability ($\psi = 1$ is exchangeable). However, ψ is not identifiable — we can only try different values as a sensitivity check.

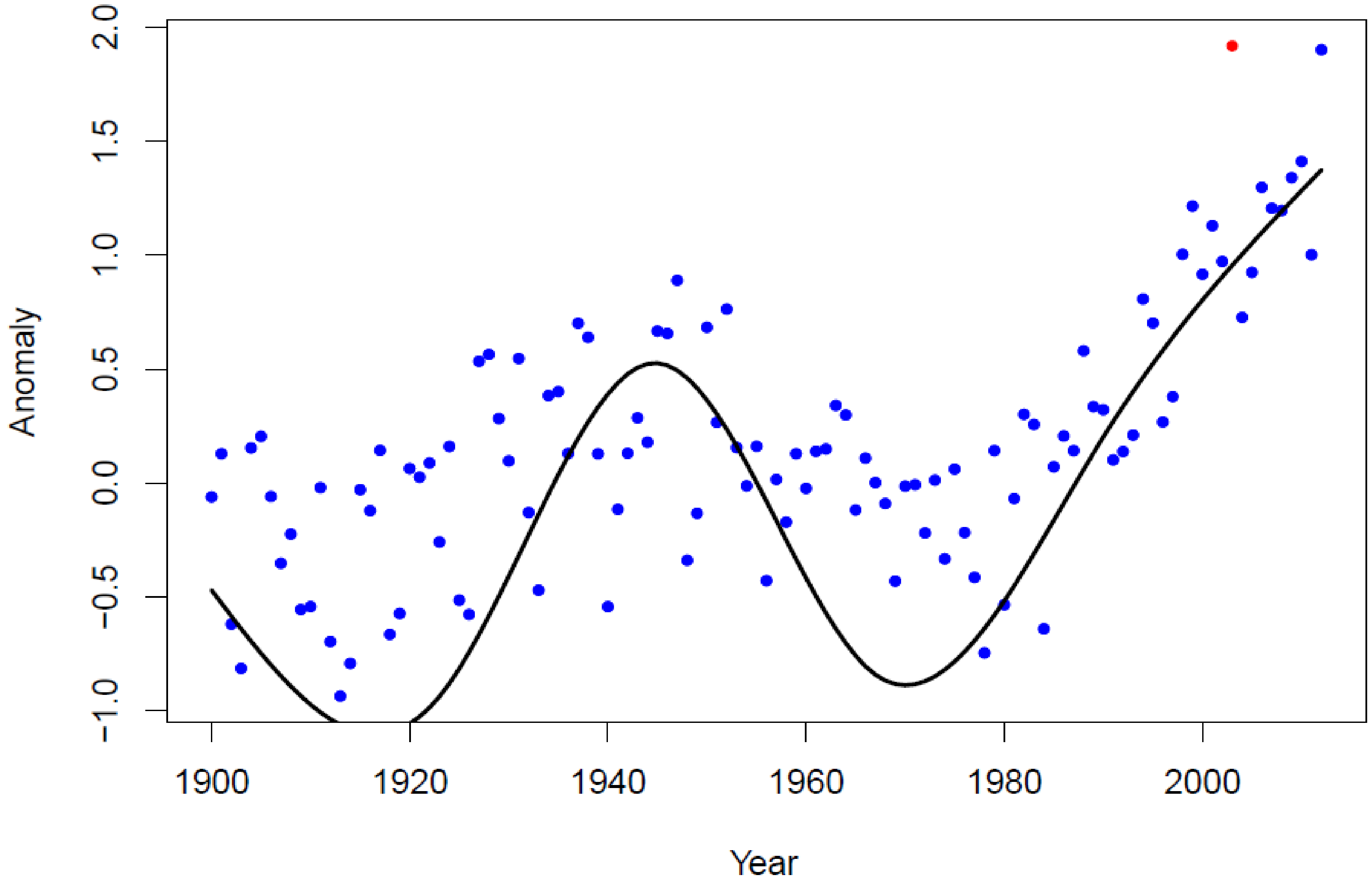
Computation

- $(M_1, D_1) \mid \theta^{(1,1)}, ..., \theta^{(1,N)} \sim WN_q(\tilde{A}, \tilde{m}, \tilde{M}^*, \tilde{F})$, where $\tilde{m} = m + N, \tilde{F} = F + N, \tilde{M}^* = \left(FM^* + \sum_{j=1}^N \theta^{(1,j)}\right)/\tilde{F}, \tilde{A} = A + FM^*M^{*T} + \sum_{j=1}^N \theta^{(j)}\theta^{(j)T} - \tilde{F}\tilde{M}^*\tilde{M}^{*T}$.
- Metropolis update for $\theta^{(1,1)}, ..., \theta^{(1,N)}$ given M_1, D_1 and $\mathbf{Y}$'s
- Metropolis update for $\theta^{(1,0)}$ based on conditional density

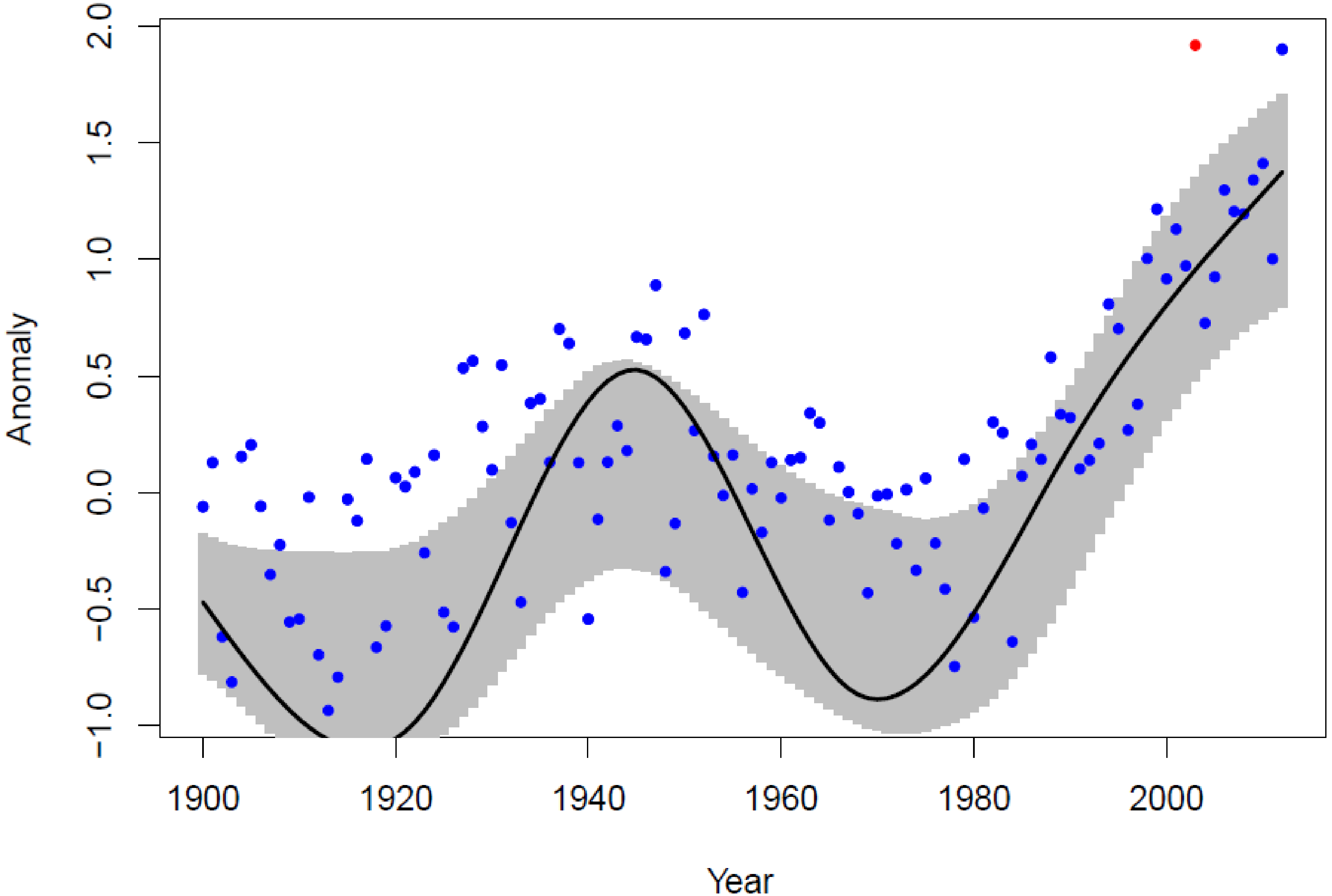
$$\exp\left\{-\frac{\psi}{2}\left(\theta^{(1,0)} - M_1\right)^T D_1\left(\theta^{(1,0)} - M_1\right)\right\} \cdot L\left(\theta^{(1,0)}; \mathbf{Y}^{(\mathrm{obs})}\right)$$

  where L is likelihood for $\theta^{(1,0)}$ given data $\mathbf{Y}^{(\mathrm{obs})}$ and $\Xi = 1$
- Similar updates for $\Xi = 0$ side of picture; up to 1,000,000 iterations

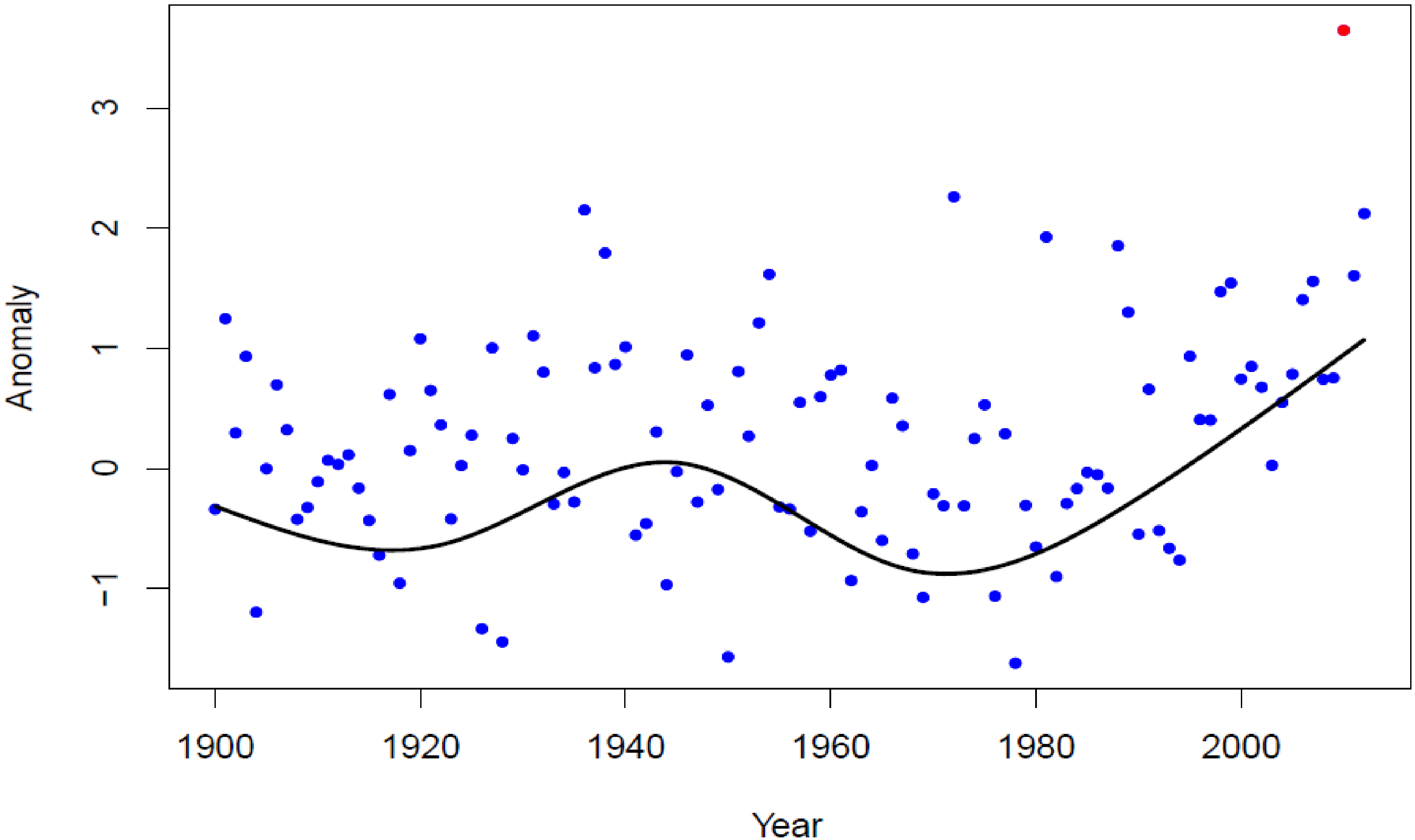
Europe Summer Mean Temperatures With Trend



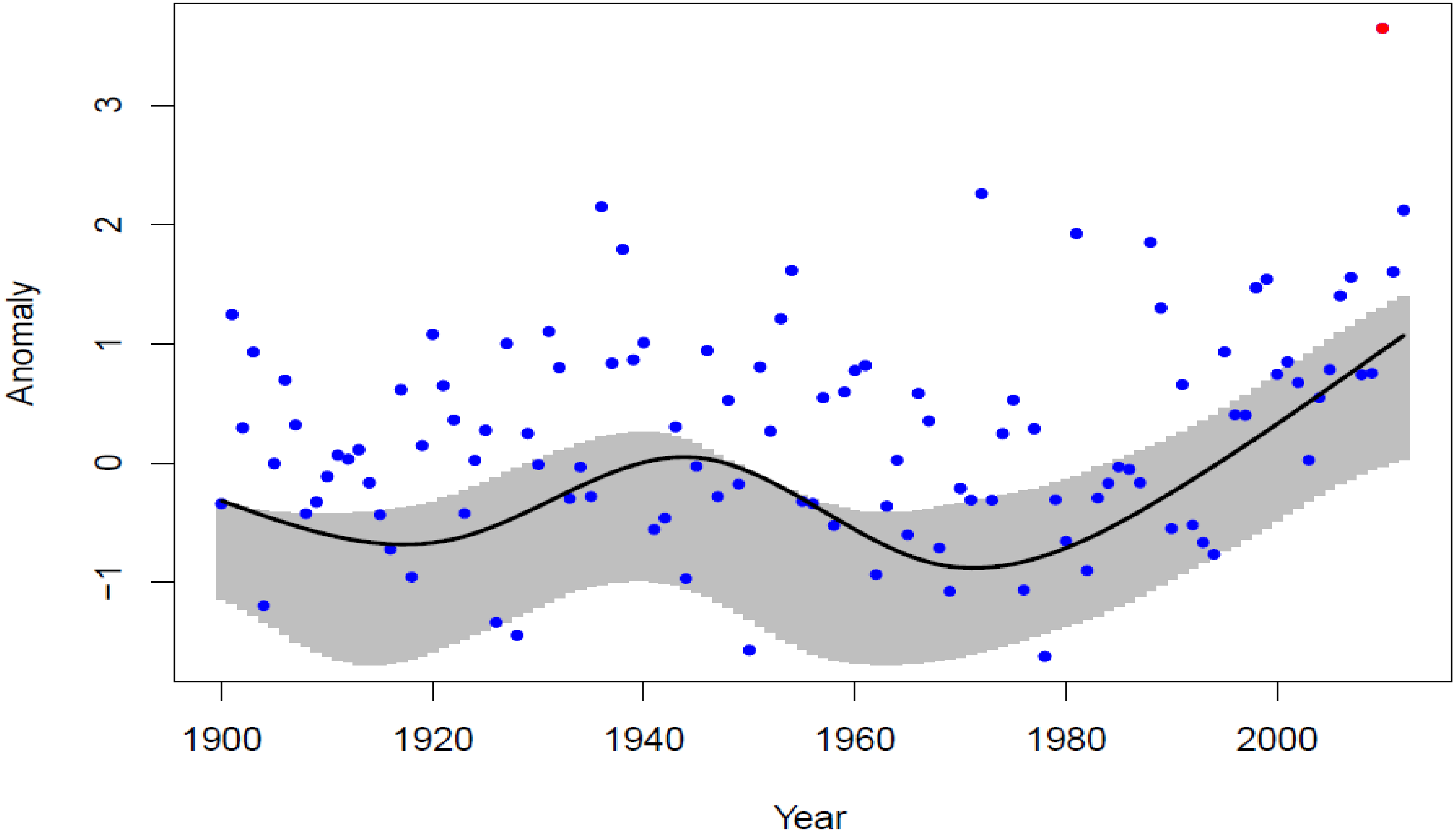
Europe Summer Mean Temperatures With Trend and Central 50% of Hierarchical Model Distribution



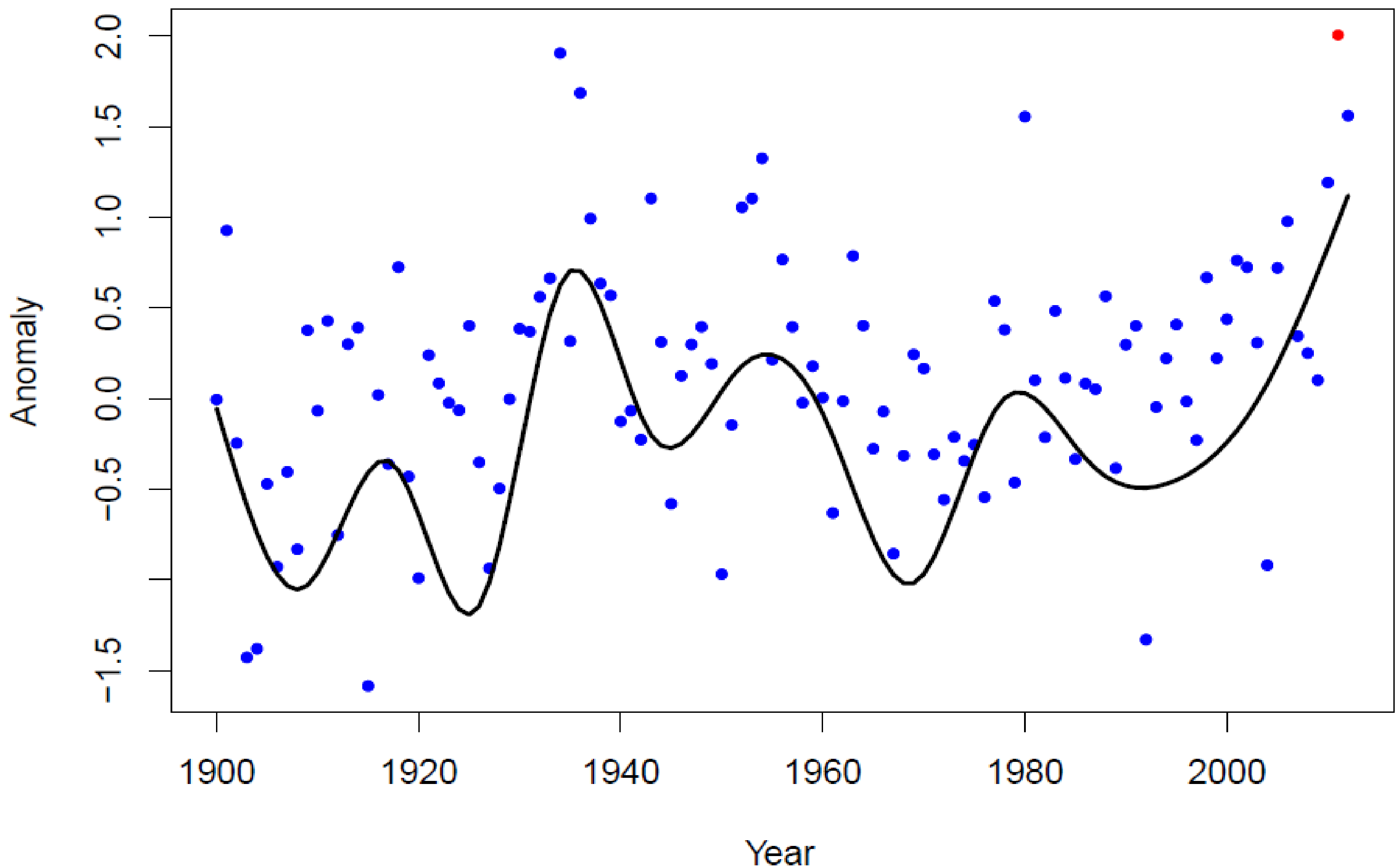
Russia Summer Mean Temperatures With Trend



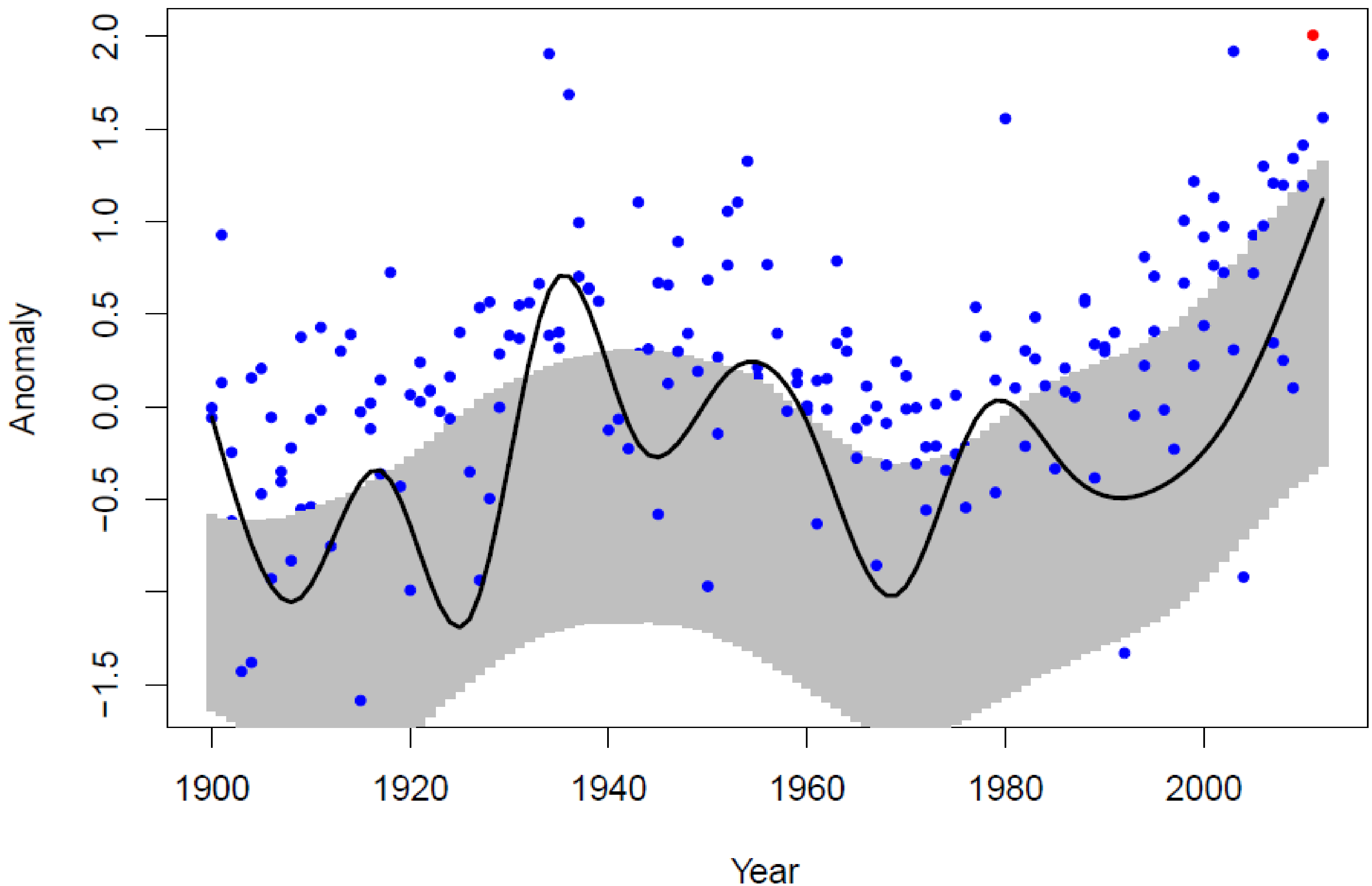
Russia Summer Mean Temperatures With Trend and Central 50% of Hierarchical Model Distribution



Central USA Summer Mean Temperatures With Trend

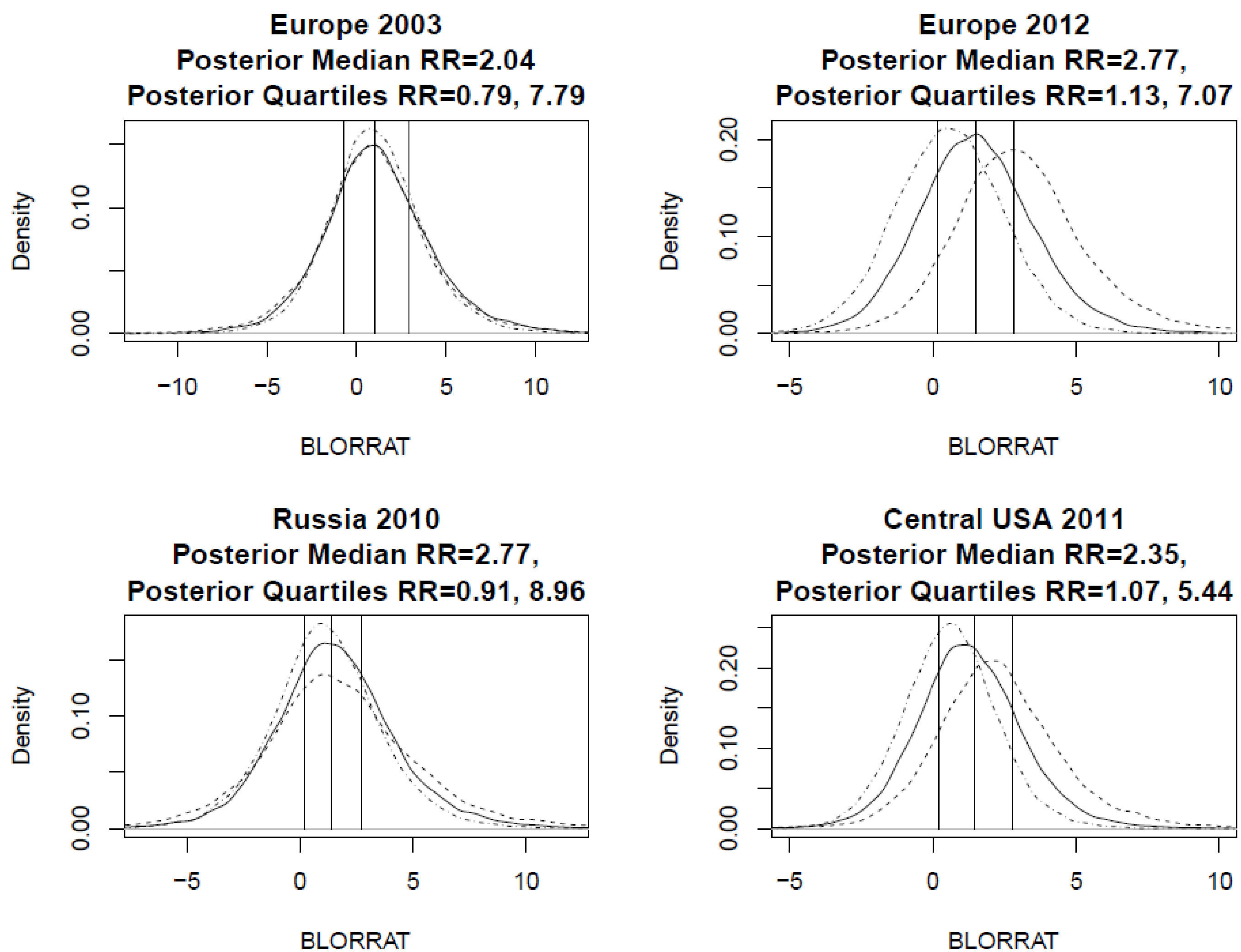


Central USA Summer Mean Temperatures With Trend and Central 50% of Hierarchical Model Distribution

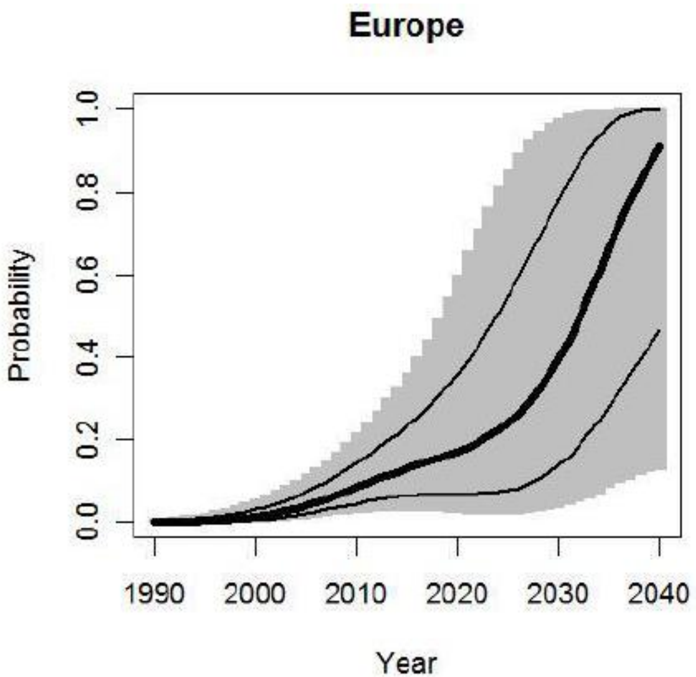


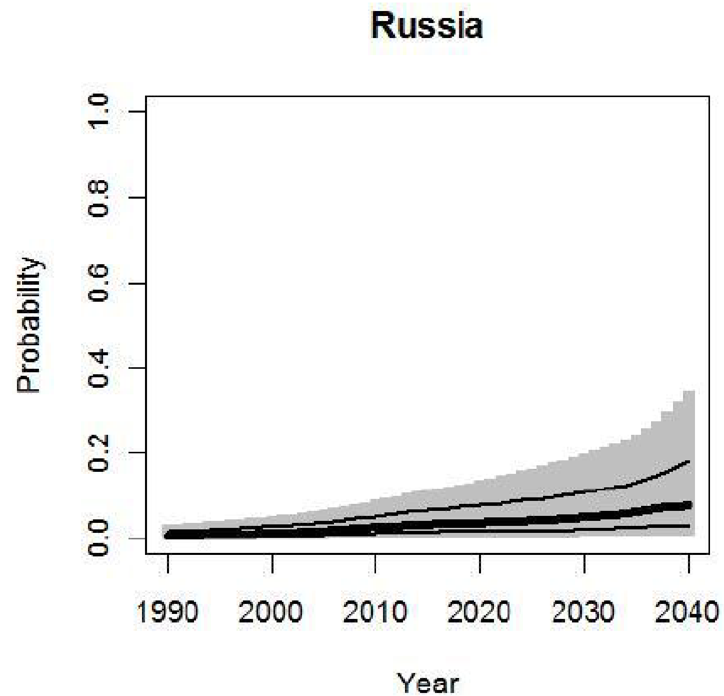
Posterior Densities for the BLORRAT

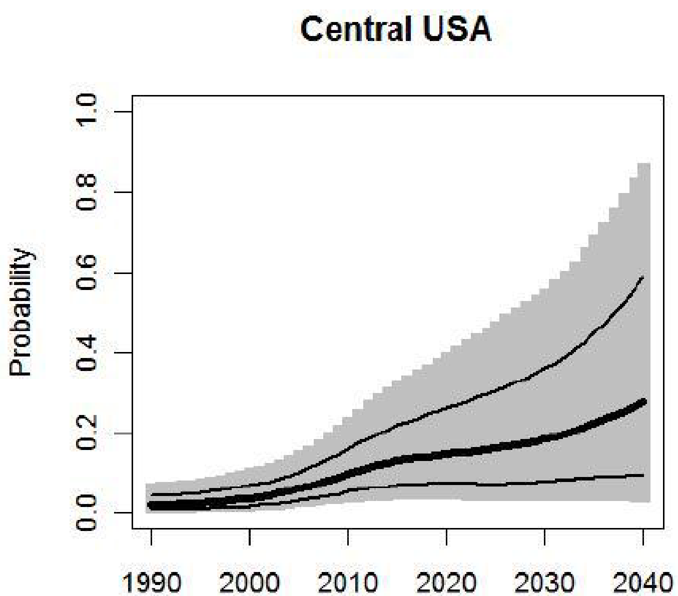
(numbers are for solid curves and equal weights; dashed curves allow for different weights between climate models and observations)



Changes in Projected Extreme Event Probabilities Over Time







Central Solid Curve: Posterior Median

Thin Outer Curves: Posterior Quartiles

Outer Limits of Shaded Region:

Posterior 10th and 90th percentiles

Conclusions

- For each of Russia 2010, Central USA 2011 and Europe 2012 events, estimated risk ratio is at least 2.3, and it's *likely* (probability at least .66) that the risk ratio is >1.5.
- We also computed future projections of extreme event probabilities; sharp increase for Europe; much less so for the other two regions studied
- Possible extension: Look at joint distributions of multiple events (e.g. extreme temperature and droughts)

JOINT DISTRIBUTIONS OF EXTREME EVENTS

Example 1. Herweijer and Seager (2008) argued that the persistence of drought patterns in various parts of the world may be explained in terms of SST patterns. One of their examples (Figure 3 of their paper) demonstrated that precipitation patterns in the south-west USA are highly correlated with those of a region of South America including parts of Uruguay and Argentina.

I computed annual precipitation means for the same regions, that show the two variables are clearly correlated (r=0.38; p¡.0001). The correlation coefficient is lower than that stated by Herweijer and Seager (r=0.57) but this is explained by their use of 6-year moving average filter, which naturally increases the correlation.

Our interest here: look at dependence in lower tail probabilities

Transform to unit Fréchet distribution (small values of precipitation corresponding to large values on Frchet scale)

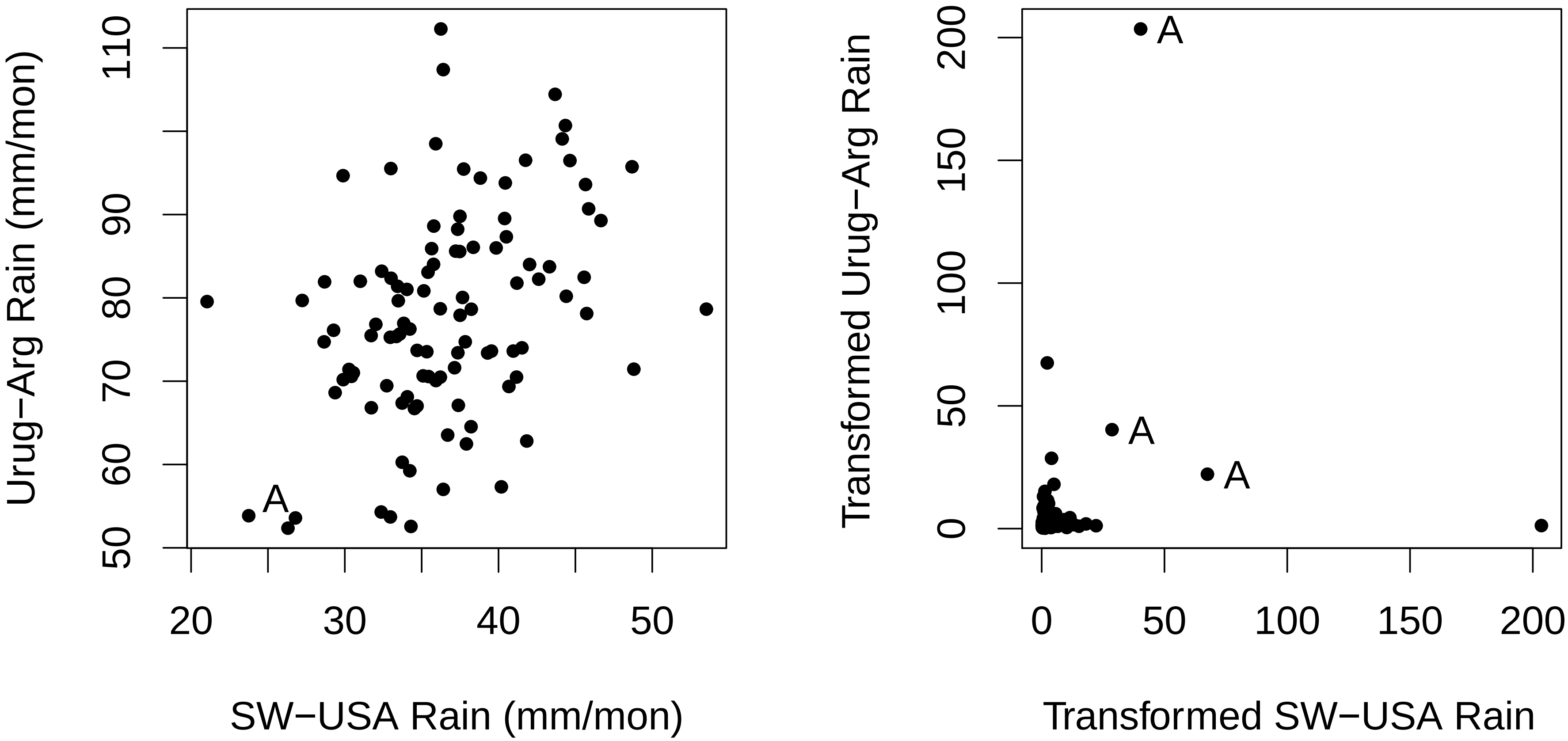


Figure 1. Left: Plot of USA annual precipitation means over latitudes 25-35°N, longitudes 95-120°W, against Argentina annual precipitation means over latitudes 30-40°S, longitudes 50-65°W, 1901-2002. Right: Same data with empirical transformation to unit Fréchet distribution. Data from gridded monthly precipitation means archived by the Climate Research Unit of the University of East Anglia.

Example 2. Lau and Kim (2012) have provided evidence that the 2010 Russian heatwave and the 2010 Pakistan floods were derived from a common set of meteorological conditions, implying a physical dependence between these very extreme events.

Using the same data source as for Example 1, I have constructed summer temperature means over Russia and precipitation means over Pakistan corresponding to the spatial areas used by Lau and Kim.

Scatterpolt of raw data and unit Fréchet transformation. 2010 value approximated — an outlier for temperature but not for precipitation.

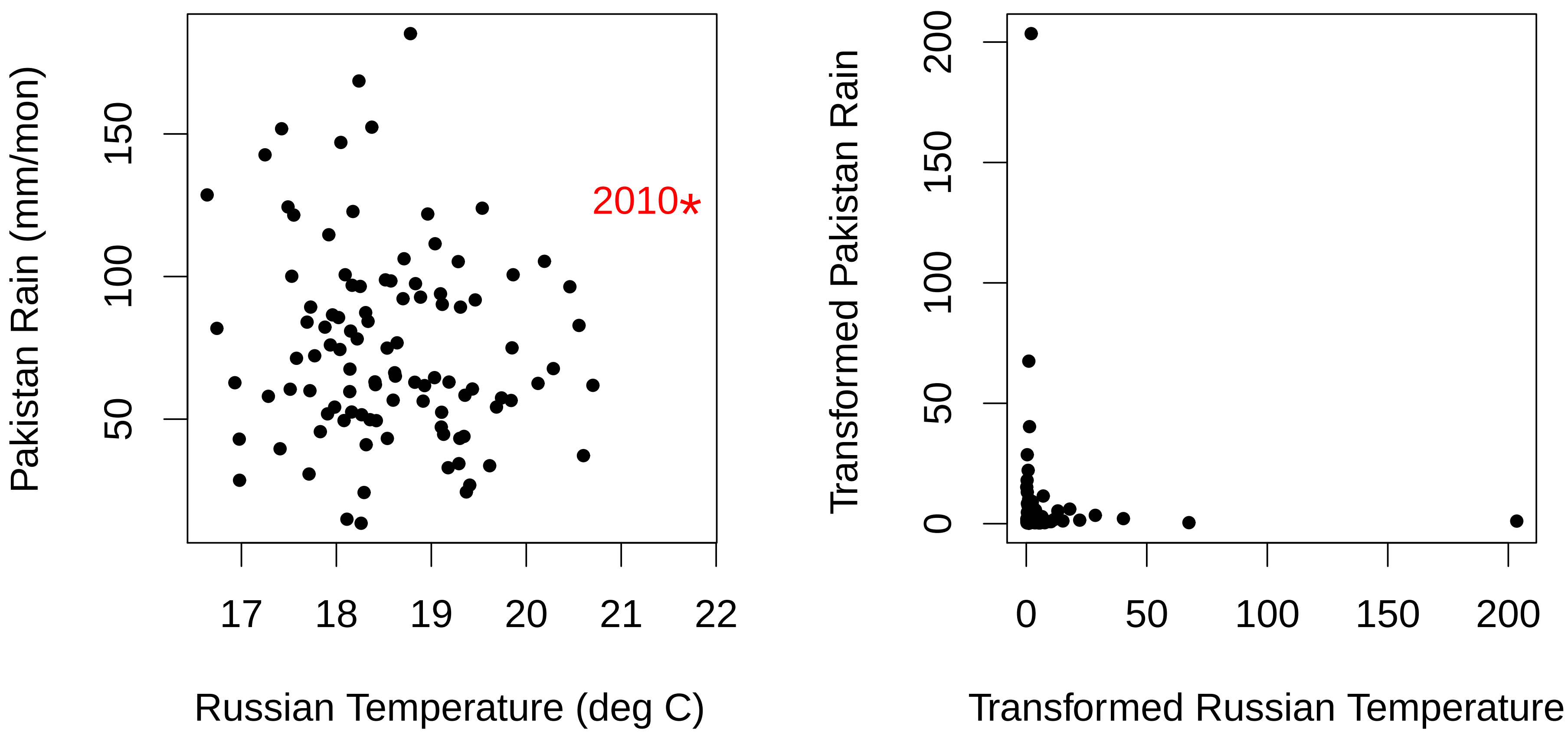


Figure 2. Left: Plot of JJA Russian temperature means against Pakistan JJA precipitation means, 1901-2002. Right: Same data with empirical transformation to unit Fréchet distribution. Data from CRU, as in Figure 1. The Russian data were averaged over 45-65°N, 30-60°E, while the Pakistan data were averaged over 32-35°N, 70-73°E, same as in Lau and Kim (2012).

Methods

Focus on the *proportion* by which the probability of a joint exceedance is greater than what would be true under independence.

Method: Fit a joint bivariate model to the exceedances above a threshold on the unit Fréchet scale

Two models:

- Classical logistic dependence model (Gumbel and Mustafi 1967; Coles and Tawn 1991)
- The η-asymmetric logistic model (Ramos and Ledford 2009)

Ref: Alexandra Ramos and Anthony Ledford (2009), A new class of models for bivariate joint tails, *J.R. Statist. Soc. B* **71**, 219-241.

3.2. Smooth H_η

The model that is detailed here is based on a modified version of the asymmetric logistic dependence structure of classical bivariate extremes (Tawn, 1988). Suppose that H_η has density h_η given by

$$h_\eta(w) = \frac{\eta - \alpha}{\alpha \eta^2 N_\rho} \left\{ (\rho w)^{-1/\alpha} + \left(\frac{1-w}{\rho} \right)^{-1/\alpha} \right\}^{\alpha/\eta - 2} \{w(1-w)\}^{-(1+1/\alpha)} \tag{3.1}$$

for $0 < w < 1$ where $N_\rho = \rho^{-1/\eta} + \rho^{1/\eta} - (\rho^{-1/\alpha} + \rho^{1/\alpha})^{\alpha/\eta}$ and $\eta \in (0, 1]$, $\alpha > 0$ and $\rho > 0$. It is straightforward to show that h_η as defined above satisfies the normalizing condition (2.5) and so, by equation (2.3), we have

$$\bar{F}_{ST}(s,t) = N_\rho^{-1} \left[(\rho s)^{-1/\eta} + \left(\frac{t}{\rho} \right)^{-1/\eta} - \left\{ (\rho s)^{-1/\alpha} + \left(\frac{t}{\rho} \right)^{-1/\alpha} \right\}^{\alpha/\eta} \right] \tag{3.2}$$

where $(s,t) \in [1, \infty) \times [1, \infty)$. The marginal survivor functions of S and T, as given by equations (2.6), have leading terms that behave like powers of s or t. For example, $\Pr(S > s)$ behaves like $s^{-1/\eta}$ if $\alpha \leqslant \eta$ and $s^{-1/\alpha}$ if $\alpha > \eta$ for large s. Thus the marginal tails of (S, T) can be heavier or of the same heaviness as the joint survivor function, depending on the relative sizes of α and η.

	Logistic Model		Ramos-Ledford Model	
	Estimate	90% CI	Estimate	90% CI
10-year	2.7	(1.2 , 4.2)	2.9	(1.2 , 5.0)
20-year	4.7	(1.4 , 7.8)	4.9	(1.2 , 9.6)
50-year	10.8	(2.1 , 18.8)	9.9	(1.4 , 23.4)

Table 1. Estimates of the increase in probability of a joint extreme event in both variables, relative to the probability under independence, for the USA/Uruguay-Argentina precipitation data. Shown are the point estimate and 90% confidence interval, under both the logistic model and the Ramos-Ledford model.

	Logistic Model		Ramos-Ledford Model	
	Estimate	90% CI	Estimate	90% CI
10-year	1.01	(1.00 , 1.01)	0.33	(0.04 , 1.4)
20-year	1.02	(1.00 , 1.03)	0.21	(0.008 , 1.8)
50-year	1.05	(1.01 , 1.07)	0.17	(0.001 , 2.9)

Table 2. Similar to Table 1, but for the Russia-Pakistan dataset.

Conclusions

- The USA–Argentina precipitation example shows clear dependence in the lower tail, though the evidence for that rests primarily on three years' data

- In contrast, the analysis of Russian temperatures and Pakistan rainfall patterns shows no historical correlation between those two variables

- Implications for future analyses: the analyses also show the merits of the Ramos-Ledford approach to bivariate extreme value modeling. The existence of a parametric family which is tractable for likelihood evaluation creates the possibility of constructing hiterarchical models for these problems.

At least three methodological extensions, all of which are topics of active research:

1. Models for multivariate extremes in > 2 dimensions

2. Spatial extremes: max-stable process, different estimation methods
   (a) Composite likelihood method
   (b) Open-faced sandwich approach
   (c) Approximations to exact likelihood, e.g. ABC method

3. Hierarchical models for bivariate and spatial extremes?